

# Open Mapping Theorems

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## Abstract

For mappings between metric linear spaces, we establish an open mapping theorem which is independent of either continuity or linearity of concerned mappings.

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The classical open mapping theorem (OMT) is the following

**Theorem A** *If  $X$  and  $Y$  are Fréchet spaces and  $f : X \rightarrow Y$  is a continuous linear operator for which  $f(X) = Y$ , then  $f$  is open.*

There are many generalizations of Theorem A [1,2,3,4,5] but these versions also deal with linear operators defined on various domain spaces. Moreover, several special propositions can imply OMT [6,7,8,9].

Every Fréchet space  $X$  is equivalent to a Hausdorff complete paranormed space  $(X, \|\cdot\|)$  [1, p.56]. Let  $U_r = \{x \in X : \|x\| < r\}$  for  $r > 0$ . It is trivial that every linear operator  $f : X \rightarrow Y$  satisfies the following

$$(p\ell 1) \quad \forall x \in X \text{ and } \varepsilon > 0 \exists \delta > 0 \text{ such that } f(x) + f(U_\delta) \subset f(x + U_\varepsilon),$$

$$(p\ell 2) \quad \forall \delta > 0 \text{ and } n \in \mathbb{N} \exists m \in \mathbb{N} \text{ such that } f(nU_\delta) \subset mf(U_\delta),$$

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(pl3)  $\forall \delta > 0 \exists \eta(\delta) > 0$  such that  $-f(U_\delta) \subset f(U_{\eta(\delta)})$  and  $\lim_{\delta \rightarrow 0} \eta(\delta) = 0$ .

It is also trivial that every continuous linear operator  $f : X \rightarrow Y$  satisfies

(pl4)  $\exists C > 0$  such that if  $\sum_{j=1}^{\infty} \|x_j\| < +\infty$  then  $\sum_{j=1}^{\infty} f(x_j) = f(z)$  with  $\|z\| \leq C \sum_{j=1}^{\infty} \|x_j\|$ .

For every Fréchet space  $X$  and topological vector space  $Y$ , the family  $PL(X, Y) = \{f \in Y^X : (pl1), (pl2), (pl3) \text{ and } (pl4) \text{ hold for } f\}$  includes all continuous linear operators and a great many of nonlinear mappings [10,11]. Hence the following recent version is a substantial improvement of Theorem A.

**Theorem B [10,11]** *Let  $X$  and  $Y$  be Hausdorff paranormed spaces. If  $f : X \rightarrow Y$  satisfies (pl1), (pl2), (pl3), (pl4) and  $f(X)$  is of second category in  $Y$ , then  $f$  is open.*

However, (pl4) implies that  $\lim_{x \rightarrow 0} f(x) = f(0) = 0$  and so the recent result Theorem B deals with mappings which are continuous at 0.

An open mapping  $f : X \rightarrow Y$  can be very complicated even for the case of  $X = Y = \mathbb{R}$ , e.g., there is an everywhere discontinuous open mapping  $f_0 : \mathbb{R} \rightarrow \mathbb{R}$  such that  $f_0(G) = \mathbb{R}$  for every nonempty open  $G \subseteq \mathbb{R}$  [12, p.168]. Hence, in this paper we will establish an open mapping theorem which is independent of either continuity or linearity of concerned mappings. Especially, we found a very large extension  $0 - pa(X, Y)$  of the family of linear operators and gave a characterization for open mappings in  $\{f \in 0 - pa(X, Y) : f \text{ is continuous at } 0\}$ .

## 1 Pseudo-additive mappings

Let  $X$  be a vector space over  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ . A function  $\|\cdot\| : X \rightarrow [0, +\infty)$  is a paranorm on  $X$  if  $\|0\| = 0$ ,  $\|-x\| = \|x\|$ ,  $\|x + z\| \leq \|x\| + \|z\|$  and  $\|t_n x_n -$

$tx\| \rightarrow 0$  whenever  $\|x_n - x\| \rightarrow 0$  and  $t_n \rightarrow t$  in  $\mathbb{K}$ . By the Kakutani theorem and the Klee theorem, every metric linear space is equivalent to a separated paranormed space, and Fréchet spaces are complete separated paranormed spaces [1, p.56].

**Definition 1.1** *Let  $X, Y$  be paranormed spaces. A mapping  $f : X \rightarrow Y$  is said to be pseudo-additive if for every  $x, z \in X$  there is  $u \in X$  with  $\|u\| \leq \|x\| + \|z\|$  such that  $f(x) + f(z) = f(u)$ , and for  $\delta > 0$  we say that  $f : X \rightarrow Y$  is  $\delta$ -pseudo-additive if for every  $x, z \in X$  with  $\|x\| + \|z\| < \delta$ ,  $f(x) + f(z) = f(u)$  with  $\|u\| \leq \|x\| + \|z\|$ .*

Let  $pa(X, Y) = \{f \in Y^X : f \text{ is pseudo-additive}\}$ ,  $\delta\text{-}pa(X, Y) = \{f \in Y^X : f \text{ is } \delta\text{-pseudo-additive}\}$ . Obviously,  $pa(X, Y) = \bigcap_{\delta > 0} \delta\text{-}pa(X, Y)$  and  $\beta\text{-}pa(X, Y) \subset \alpha\text{-}pa(X, Y)$  whenever  $0 < \alpha < \beta$ . Let  $0\text{-}pa(X, Y) = \bigcup_{\delta > 0} \delta\text{-}pa(X, Y)$ .

Henceforth,  $X$  and  $Y$  are nontrivial Hausdorff (i.e., separated) paranormed spaces over  $\mathbb{K}$ ,  $\mathcal{N}(X)$  (respectively,  $\mathcal{N}(Y)$ ) is the family of neighborhoods of 0 in  $X$  (respectively,  $Y$ ) and  $U_r = \{x \in X : \|x\| < r\}$ ,  $V_r = \{y \in Y : \|y\| < r\}$  for  $r > 0$ .

**Proposition 1.1** *Let  $h : X \rightarrow Y$  be arbitrary. For  $\delta > 0$  and  $g \in pa(X, Y)$ , let*

$$f(x) = \begin{cases} g(x), & \|x\| < \delta, \\ h(x), & \|x\| \geq \delta, \end{cases}$$

*then  $f \in \delta\text{-}pa(X, Y)$ .*

Proof. If  $x, z \in X$ ,  $\|x\| + \|z\| < \delta$ , then  $f(x) + f(z) = g(x) + g(z) = g(u)$  with  $\|u\| \leq \|x\| + \|z\| < \delta$  and so  $f(x) + f(z) = f(u)$ .  $\square$

Thus,  $0\text{-}pa(X, Y) \setminus pa(X, Y)$  is a very large family.

**Proposition 1.2** *If  $f \in 0\text{-}pa(X, Y)$ , then  $f(0) = 0$ .*

Proof.  $f \in \delta\text{-}pa(X, Y)$  for some  $\delta > 0$  and  $f(0) + f(0) = f(u)$  with  $\|u\| \leq \|0\| + \|0\| = 0$ . Since  $X$  is Hausdorff,  $u = 0$  and  $f(0) + f(0) = f(0)$ ,  $f(0) = 0$ .

□

**Proposition 1.3** *If  $f \in pa(X, Y)$ , then for every  $n \in \mathbb{N}$  and every  $x_1, x_2, \dots, x_n \in X$  there is  $u \in X$  with  $\|u\| \leq \sum_{j=1}^n \|x_j\|$  such that  $\sum_{j=1}^n f(x_j) = f(u)$ .*

Proof.  $f(x_1) + f(x_2) = f(u_2)$  with  $\|u_2\| \leq \|x_1\| + \|x_2\|$ . If  $2 \leq k < n$  and  $\sum_{j=1}^k f(x_j) = f(u_k)$  with  $\|u_k\| \leq \sum_{j=1}^k \|x_j\|$ , then  $\sum_{j=1}^{k+1} f(x_j) = f(u_k) + f(x_{k+1}) = f(u_{k+1})$  with  $\|u_{k+1}\| \leq \|u_k\| + \|x_{k+1}\| \leq \sum_{j=1}^{k+1} \|x_j\|$ . □

**Proposition 1.4** *If  $\delta > 0$  and  $f \in \delta\text{-}pa(X, Y)$ , then for every  $\{x_j\} \subset X$  with  $\sum_{j=1}^\infty \|x_j\| < \delta$  and  $n \in \mathbb{N}$  there is  $u_n \in X$  such that  $\|u_n\| \leq \sum_{j=1}^n \|x_j\|$  and  $\sum_{j=1}^n f(x_j) = f(u_n)$ .*

Proof.  $\sum_{j=1}^2 f(x_j) = f(u_2)$  with  $\|u_2\| \leq \sum_{j=1}^2 \|x_j\|$ . If  $k \geq 2$  and  $\sum_{j=1}^k f(x_j) = f(u_k)$  with  $\|u_k\| \leq \sum_{j=1}^k \|x_j\|$ , then  $\|u_k\| + \|x_{k+1}\| \leq \sum_{j=1}^{k+1} \|x_j\| \leq \sum_{j=1}^\infty \|x_j\| < \delta$  and so  $\sum_{j=1}^{k+1} f(x_j) = f(u_k) + f(x_{k+1}) = f(u_{k+1})$  with  $\|u_{k+1}\| \leq \|u_k\| + \|x_{k+1}\| \leq \sum_{j=1}^{k+1} \|x_j\|$ . □

**Example 1.1** *Let  $(X, \|\cdot\|)$  be a normed space. Then  $\|\cdot\|^p \in pa(X, \mathbb{R})$  for all  $p \geq 1$ . To this end pick an  $x_0 \in X$  with  $\|x_0\| = 1$ , e.g.,  $x_0 = \frac{1}{\|x\|}x$  where  $x \neq 0$ . If  $p \geq 1$  then  $\|x\|^p + \|z\|^p = \|(\|x\|^p + \|z\|^p)^{1/p} x_0\|^p$  with  $\|(\|x\|^p + \|z\|^p)^{1/p} x_0\| = (\|x\|^p + \|z\|^p)^{1/p} \leq \|x\| + \|z\|$ .*

**Proposition 1.5** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuous. Then  $f \in pa(\mathbb{R}, \mathbb{R})$  if and only if  $f(0) = 0$  and*

$$\min_{|t| \leq a+b} f(t) \leq \min_{|t| \leq a} f(t) + \min_{|t| \leq b} f(t) \leq \max_{|t| \leq a} f(t) + \max_{|t| \leq b} f(t) \leq \max_{|t| \leq a+b} f(t), \quad \forall a, b > 0.$$

Proof.  $\implies$  By Proposition 1.2,  $f(0) = 0$ . Let  $a, b > 0$ . There are  $x_1, x_2 \in [-a, a]$  and  $z_1, z_2 \in [-b, b]$  such that  $f(x_1) = \min_{|t| \leq a} f(t)$ ,  $f(x_2) = \max_{|t| \leq a} f(t)$ ,  $f(z_1) = \min_{|t| \leq b} f(t)$  and  $f(z_2) = \max_{|t| \leq b} f(t)$ . Then  $f(x_1) + f(z_1) = f(u)$  with  $|u| \leq |x_1| + |z_1| \leq a + b$ ,  $f(x_2) + f(z_2) = f(v)$  with  $|v| \leq |x_2| + |z_2| \leq a + b$  and so  $\min_{|t| \leq a+b} f(t) \leq f(u) = f(x_1) + f(z_1) = \min_{|t| \leq a} f(t) + \min_{|t| \leq b} f(t) \leq \max_{|t| \leq a} f(t) + \max_{|t| \leq b} f(t) = f(v) \leq \max_{|t| \leq a+b} f(t)$ .

$\Leftarrow$  Let  $x, z \in \mathbb{R} \setminus \{0\}$ . Since  $\min_{|t| \leq |x|+|z|} f(t) = f(v)$  with  $|v| \leq |x| + |z|$  and  $\max_{|t| \leq |x|+|z|} f(t) = f(w)$  with  $|w| \leq |x| + |z|$ ,  $f(v) \leq \min_{|t| \leq |x|} f(t) + \min_{|t| \leq |z|} f(t) \leq f(x) + f(z) \leq \max_{|t| \leq |x|} f(t) + \max_{|t| \leq |z|} f(t) \leq f(w)$ . Say that  $w \leq v$ . Then  $f(x) + f(z) = f(u)$  for some  $u \in [w, v]$  and  $|u| \leq \max(|w|, |v|) \leq |x| + |z|$ .  $\square$

Now we can show that  $pa(\mathbb{R}, \mathbb{R})$  includes a great many of nonlinear continuous functions.

**Example 1.2** For  $x \in \mathbb{R}$  let  $f(x) = \sinh x = (e^x - e^{-x})/2$ ,  $g(x) = x^n$ ,

$$h(x) = \begin{cases} e^x - 1, & x \geq 0, \\ -\tan \frac{\pi x^2}{2(x^2+1)}, & x < 0, \end{cases} \quad \xi(x) = \begin{cases} \alpha x, & x \geq 0, \\ \beta x, & x < 0 \end{cases} \quad (0 < \alpha < \beta).$$

Then  $f, g, h, \xi \in pa(\mathbb{R}, \mathbb{R})$ .

**Example 1.3** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a strictly monotone continuous function. If  $f(0) = 0$  and  $|f(\cdot)|$  is convex, then  $f \in pa(\mathbb{R}, \mathbb{R})$ .

However, pseudo-additivity is independent of continuity.

**Example 1.4** (1) Let  $f_0 : \mathbb{R} \rightarrow \mathbb{R}$  be the function in [12, p.168].  $f_0$  is an everywhere discontinuous open mapping such that  $f_0(0) = f_0(n) = f_0(-n) = 0$  for all  $n \in \mathbb{N}$  and  $f_0(G) = \mathbb{R}$  for every nonempty open  $G \subseteq \mathbb{R}$ . For  $z \in \mathbb{R}$ ,  $f_0(0) + f_0(z) = f(z)$ ,  $|z| = 0 + |z| = |0| + |z|$ . If  $x, z \in \mathbb{R}$ ,  $x \neq 0$ , then  $f_0(x) + f_0(z) \in \mathbb{R} = f_0[(0, |x|)]$  and so  $f_0(x) + f_0(z) = f(u)$  where  $0 < u = |u| < |x| \leq |x| + |z|$ . Thus,  $f_0 \in pa(\mathbb{R}, \mathbb{R})$ .

(2) Let

$$f(x) = \begin{cases} 1/|x|, & x \in \mathbb{Q} \setminus \{0\}, \\ 0, & x \in \{0\} \cup (\mathbb{R} \setminus \mathbb{Q}). \end{cases}$$

Then  $f$  is everywhere discontinuous but  $f \in pa(\mathbb{R}, \mathbb{R})$ . In fact, for  $x \in \{0\} \cup (\mathbb{R} \setminus \mathbb{Q})$ ,  $f(x) + f(z) = f(z)$  where  $|z| \leq |x| + |z|$ , and for  $x, z \in \mathbb{Q} \setminus \{0\}$  we have  $f(x) + f(z) = \frac{1}{|x|} + \frac{1}{|z|} = \frac{|x|+|z|}{|xz|} = f(\frac{|xz|}{|x|+|z|})$  where  $\frac{|xz|}{|x|+|z|} \leq |x| + |z|$ .

(3) Let

$$f(x) = \begin{cases} x, & x \in Q, \\ 0, & x \in \mathbb{R} \setminus Q. \end{cases}$$

Then  $f$  is discontinuous at each  $x \neq 0$  but  $f \in pa(\mathbb{R}, \mathbb{R})$ .

**Proposition 1.6** Let  $(Y, \|\cdot\|)$  be a normed space. If  $f \in \delta\text{-}pa(X, Y)$  is discontinuous at 0, then  $\sup_{\|x\| < \varepsilon} \|f(x)\| = +\infty$  for all  $\varepsilon > 0$ . In particular, if a linear  $f : X \rightarrow Y$  is not continuous, then  $\sup_{\|x\| < \varepsilon} \|f(x)\| = +\infty$  for all  $\varepsilon > 0$ .

Proof. There is an  $\alpha > 0$  and  $\{z_k\} \subset X$  such that  $z_k \rightarrow 0$  but  $\|f(z_k)\| \geq \alpha$  for all  $k$ . Then there exist integers  $k_1 < k_2 < \dots$  such that  $\|z_{k_n}\| < \delta/n$  for all  $n$ . Letting  $x_n = z_{k_n}$ ,  $\|x_n\| < \delta/n$  and  $\|f(x_n)\| = \|f(z_{k_n})\| \geq \alpha$ ,  $n = 1, 2, 3, \dots$ .

There is  $u_1 \in X$  such that  $\|u_1\| \leq \|x_2\| + \|x_2\| < \delta$  and  $f(u_1) = f(x_2) + f(x_2) = 2f(x_2)$ ,  $\|f(u_1)\| = 2\|f(x_2)\| \geq 2\alpha$ . Since  $4\|x_8\| < \frac{\delta}{2} < \delta$ , Proposition 1.4 shows that  $4f(x_8) = f(u_2)$  with  $\|u_2\| \leq 4\|x_8\| < \delta/2$ . In this way, we have  $\{u_k\} \subset X$  for which  $\|u_k\| \leq 2^k\|x_{2^{2k-1}}\| < \delta/2^{k-1}$  and  $\|f(u_k)\| = 2^k\|f(x_{2^{2k-1}})\| \geq 2^k\alpha$  for all  $k$ . Then  $\|u_k\| \rightarrow 0$  and  $\|f(u_k)\| \rightarrow +\infty$ .  $\square$

**Proposition 1.7** If  $\varphi \in pa(\mathbb{R}, \mathbb{R})$ ,  $i = \sqrt{-1}$  and

$$f(a + bi) = \varphi(a) + \varphi(b)i, \quad \forall a, b \in \mathbb{R},$$

then  $f \in pa(\mathbb{C}, \mathbb{C})$ . If, in addition,  $\varphi$  is a homeomorphism from  $\mathbb{R}$  onto  $\mathbb{R}$ , then  $f$  is a homeomorphism from  $\mathbb{C}$  onto  $\mathbb{C}$ .

Proof. For  $a_1, a_2, b_1, b_2 \in \mathbb{R}$ ,  $f(a_1 + a_2i) + f(b_1 + b_2i) = \varphi(a_1) + \varphi(a_2)i + \varphi(b_1) + \varphi(b_2)i = \varphi(c) + \varphi(d)i = f(c + di)$  where  $|c| \leq |a_1| + |b_1|$ ,  $|d| \leq |a_2| + |b_2|$ . Then  $|c + di| = \sqrt{c^2 + d^2} \leq \sqrt{(|a_1| + |b_1|)^2 + (|a_2| + |b_2|)^2} \leq \sqrt{a_1^2 + a_2^2} + \sqrt{b_1^2 + b_2^2} = |a_1 + a_2i| + |b_1 + b_2i|$ ,  $f \in pa(\mathbb{C}, \mathbb{C})$ .

Suppose that  $\varphi$  is a homeomorphism from  $\mathbb{R}$  onto  $\mathbb{R}$ . If  $a + bi \neq c + di$ , then  $a \neq c$  or  $b \neq d$  and so  $\varphi(a) \neq \varphi(c)$  or  $\varphi(b) \neq \varphi(d)$  and so  $f(a + bi) = \varphi(a) + \varphi(b)i \neq \varphi(c) + \varphi(d)i = f(c + di)$ ,  $f$  is one to one. If  $a, b \in \mathbb{R}$ , then  $a + bi = \varphi(\varphi^{-1}(a)) + \varphi(\varphi^{-1}(b))i = f[\varphi^{-1}(a) + \varphi^{-1}(b)i]$  and so  $f(\mathbb{C}) = \mathbb{C}$ .

Let  $a, b, a_n, b_n \in \mathbb{R}$  for  $n \in \mathbb{N}$ . If  $a_n + b_n i \rightarrow a + bi$  in  $\mathbb{C}$ , then  $a_n \rightarrow a$ ,  $b_n \rightarrow b$  and so  $f(a_n + b_n i) = \varphi(a_n) + \varphi(b_n)i \rightarrow \varphi(a) + \varphi(b)i = f(a + bi)$ ,  $f^{-1}(a_n + b_n i) = \varphi^{-1}(a_n) + \varphi^{-1}(b_n)i \rightarrow \varphi^{-1}(a) + \varphi^{-1}(b)i = f^{-1}(a + bi)$ .  $\square$

Note that  $pa(\mathbb{R}, \mathbb{R})$  includes a great many of nonlinear homeomorphisms from  $\mathbb{R}$  onto  $\mathbb{R}$ , e.g., the functions in Example 1.2 and 1.3. By Proposition 1.7,  $pa(\mathbb{C}, \mathbb{C})$  also includes nonlinear homeomorphisms as many as  $pa(\mathbb{R}, \mathbb{R})$ , at least.

**Proposition 1.8** *Let  $\Omega$  be a compact Hausdorff space and  $C(\Omega) = \{\xi \in \mathbb{K}^\Omega : \xi \text{ is continuous}\}$ . Let  $A : C(\Omega) \rightarrow Y$  be additive, i.e.,  $A(\xi) + A(\eta) = A(\xi + \eta)$  for  $\xi, \eta \in C(\Omega)$ . If  $\varphi \in pa(\mathbb{K}, \mathbb{K})$  is a homeomorphism from  $\mathbb{K}$  onto  $\mathbb{K}$ , and*

$$f_{A,\varphi}(\xi) = A(\varphi \circ \xi), \quad \xi \in C(\Omega),$$

*then  $f_{A,\varphi} \in pa(C(\Omega), Y)$ .*

Proof. Let  $\xi, \eta \in C(\Omega)$  and  $\zeta = \varphi^{-1} \circ (\varphi \circ \xi + \varphi \circ \eta)$ . Then  $\zeta \in C(\Omega)$ . For  $\omega \in \Omega$ ,  $\varphi(\zeta(\omega)) = \varphi(\xi(\omega)) + \varphi(\eta(\omega)) = \varphi(u)$  with  $|u| \leq |\xi(\omega)| + |\eta(\omega)|$  and  $u = \zeta(\omega)$  since  $\varphi$  is one to one. Then  $\|\zeta\| = \sup_{\omega \in \Omega} |\zeta(\omega)| \leq \sup_{\omega \in \Omega} (|\xi(\omega)| + |\eta(\omega)|) \leq \sup_{\omega \in \Omega} |\xi(\omega)| + \sup_{\omega \in \Omega} |\eta(\omega)| = \|\xi\| + \|\eta\|$ , and

$$f_{A,\varphi}(\xi) + f_{A,\varphi}(\eta) = A(\varphi \circ \xi) + A(\varphi \circ \eta) = A(\varphi \circ \xi + \varphi \circ \eta) = A(\varphi \circ \zeta) = f_{A,\varphi}(\zeta).$$

$\square$

Thus,  $pa(C(\Omega), Y)$  includes a great many of nonlinear mappings.

**Proposition 1.9** *If  $\varphi \in pa(\mathbb{R}, \mathbb{R})$  is continuous and one to one,  $f(a + bi) = \varphi(a) + \varphi(b)i$  for  $a, b \in \mathbb{R}$ , then there is an  $\varepsilon > 0$  such that*

$$\sup_{0 < |t| < \varepsilon} \left| \frac{\varphi(t)}{t} \right| < +\infty, \quad \sup_{0 < |w| < \varepsilon} \left| \frac{f(w)}{w} \right| < +\infty.$$

Proof. By Proposition 1.2,  $\varphi(0) = 0$ . Since  $\varphi$  is continuous and one to one,  $\varphi$  is strictly monotonic. Say that  $\varphi$  is strictly increasing. Then  $\varphi$  is differentiable almost everywhere and so there exist  $a < 0$  and  $b > 0$  such that  $0 < \varphi'(a) <$

$+\infty$  and  $0 < \varphi'(b) < +\infty$ . Pick an  $\varepsilon > 0$  such that

$$0 < \frac{\varphi(b+t) - \varphi(b)}{t} < \varphi'(b)+1, \quad 0 < \frac{\varphi(a-t) - \varphi(a)}{-t} < \varphi'(a)+1, \quad \forall 0 < t < \varepsilon.$$

By Proposition 1.5, if  $0 < t < \varepsilon$  then  $\varphi(t) + \varphi(b) \leq \varphi(b+t)$  and  $\varphi(a-t) \leq \varphi(a) + \varphi(-t)$ , i.e., if  $0 < t < \varepsilon$  then

$$0 < \frac{\varphi(t)}{t} \leq \frac{\varphi(b+t) - \varphi(b)}{t} < \varphi'(b)+1, \quad 0 < \frac{\varphi(-t)}{-t} \leq \frac{\varphi(a-t) - \varphi(a)}{-t} < \varphi'(a)+1.$$

Then  $0 < \sup_{0 < |t| < \varepsilon} \left| \frac{\varphi(t)}{t} \right| = \sup_{0 < |t| < \varepsilon} \frac{\varphi(t)}{t} \leq \varphi'(a) + \varphi'(b) + 1 < +\infty$ .

Since  $\varphi(0) = 0$ ,  $\sup_{0 < |t+si| < \varepsilon} \left| \frac{f(t+si)}{t+si} \right| = \sup_{0 < \sqrt{t^2+s^2} < \varepsilon} \frac{\sqrt{[\varphi(t)]^2 + [\varphi(s)]^2}}{\sqrt{t^2+s^2}} \leq \sup_{0 < \sqrt{t^2+s^2} < \varepsilon} \frac{|\varphi(t)| + |\varphi(s)|}{\sqrt{t^2+s^2}} \leq 2 \sup_{0 < |t| < \varepsilon} \left| \frac{\varphi(t)}{t} \right| \leq 2[\varphi'(a) + \varphi'(b) + 1] < +\infty$ .  $\square$

We now consider sequence spaces such as  $\ell^p = \{(t_j) \in \mathbb{K}^{\mathbb{N}} : \sum_{j=1}^{\infty} |t_j|^p < +\infty\}$ , etc.

**Proposition 1.10** *Let  $n \in \mathbb{N}$ ,  $p > 0$ ,  $E \in \{\mathbb{K}^n, \ell^p, c_{00}, c_0, c, \ell^\infty\}$  and  $A : E \rightarrow Y$  be additive. Let  $\varphi \in pa(\mathbb{K}, \mathbb{K})$  be a homeomorphism from  $\mathbb{K}$  onto  $\mathbb{K}$  such that if  $\mathbb{K} = \mathbb{C}$  then  $\varphi(a+bi) = \varphi(a) + \varphi(b)i$  where  $\varphi|_{\mathbb{R}}$  is a homeomorphism from  $\mathbb{R}$  onto  $\mathbb{R}$ . If*

$$f_{A,\varphi}(\{t_j\}) = A(\{\varphi(t_j)\}), \quad \{t_j\} \in E,$$

*then  $f_{A,\varphi} \in pa(E, Y)$ .*

*Proof.* Since  $\varphi$  is continuous and  $\varphi(0) = 0$  by Proposition 1.2, if  $E \in \{\mathbb{K}^n, c_{00}, c_0, c, \ell^\infty\}$  then  $\{\varphi(t_j)\} \in E$  for all  $\{t_j\} \in E$ . Let  $\{t_j\} \in \ell^p$ . By Proposition 1.9, there is an  $\varepsilon > 0$  for which  $M = \sup_{0 < |w| < \varepsilon} \left| \frac{\varphi(w)}{w} \right| < +\infty$ . Since  $\varphi(0) = 0$ ,  $|\varphi(w)| \leq M|w|$  for all  $|w| < \varepsilon$ , it follows from  $t_j \rightarrow 0$  that there is  $j_0 \in \mathbb{N}$  such that  $\sum_{j=1}^{\infty} |\varphi(t_j)|^p \leq \sum_{j=1}^{j_0} |\varphi(t_j)|^p + M^p \sum_{j > j_0} |t_j|^p < +\infty$ , i.e.,  $\{\varphi(t_j)\} \in \ell^p$ .

Let  $\{t_j\}, \{s_j\} \in E$ . Then  $\varphi(t_j) + \varphi(s_j) = \varphi(r_j)$  with  $|r_j| \leq |t_j| + |s_j|$  for all  $j$ . If  $E = \ell^p$ , then  $\{|t_j| + |s_j|\} \in \ell^p$  and so  $\{\varphi^{-1}[\varphi(t_j) + \varphi(s_j)]\} = \{r_j\} \in \ell^p$ . If  $E =$



$c$ , then  $\lim_j \varphi^{-1}[\varphi(t_j) + \varphi(s_j)] = \varphi^{-1}[\lim_j \varphi(t_j) + \lim_j \varphi(s_j)] = \varphi^{-1}[\varphi(\lim_j t_j) + \varphi(\lim_j s_j)]$  exists, i.e.,  $\{\varphi^{-1}[\varphi(t_j) + \varphi(s_j)]\} = \{r_j\} \in c$ . Clearly, for the case of  $E \in \{\mathbb{K}^n, c_{00}, c_0, \ell^\infty\}$  we also have that  $\{\varphi^{-1}[\varphi(t_j) + \varphi(s_j)]\} = \{r_j\} \in E$ . Then  $f_{A,\varphi}(\{t_j\}) + f_{A,\varphi}(\{s_j\}) = A(\{\varphi(t_j)\}) + A(\{\varphi(s_j)\}) = A(\{\varphi(t_j) + \varphi(s_j)\}) = A(\{\varphi(r_j)\}) = f_{A,\varphi}(\{r_j\})$ .

Observe that  $|r_j| \leq |t_j| + |s_j|$  for all  $j$ . If  $E = \ell^\infty$ , then

$$\|\{r_j\}\| = \sup_j |r_j| \leq \sup_j (|t_j| + |s_j|) \leq \sup_j |t_j| + \sup_j |s_j| = \|\{t_j\}\| + \|\{s_j\}\|;$$

if  $E = \ell^p$ ,  $0 < p < 1$ , then

$$\|\{r_j\}\| = \sum_{j=1}^{\infty} |r_j|^p \leq \sum_{j=1}^{\infty} (|t_j| + |s_j|)^p \leq \sum_{j=1}^{\infty} |t_j|^p + \sum_{j=1}^{\infty} |s_j|^p = \|\{t_j\}\| + \|\{s_j\}\|;$$

if  $E = \ell^p$ ,  $p \geq 1$ , then

$$\begin{aligned} \|\{r_j\}\| &= \left( \sum_{j=1}^{\infty} |r_j|^p \right)^{1/p} \leq \left( \sum_{j=1}^{\infty} (|t_j| + |s_j|)^p \right)^{1/p} \\ &\leq \left( \sum_{j=1}^{\infty} |t_j|^p \right)^{1/p} + \left( \sum_{j=1}^{\infty} |s_j|^p \right)^{1/p} = \|\{t_j\}\| + \|\{s_j\}\|. \end{aligned}$$

Thus,  $f_{A,\varphi} \in pa(E, Y)$ . □

## 2 Open mapping theorem

Let  $(X, \|\cdot\|)$  and  $(Y, \|\cdot\|)$  be paranormed spaces and  $U_r = \{x \in X : \|x\| < r\}$ ,  $V_r = \{y \in Y : \|y\| < r\}$  for  $r > 0$ . Let  $\mathcal{N}(Y)$  be the family of neighborhoods of  $0 \in Y$ .

The classical OMT (Theorem A ) can be rewrote to the following more general version [1, p.58].

**Theorem A'** *Let  $X$  be a Fréchet space and  $Y$  a Hausdorff paranormed space.*

*If  $f : X \rightarrow Y$  is a continuous linear operator such that*

$$(O1) \quad \overline{f(U_r)} \in \mathcal{N}(Y), \quad \forall r > 0,$$

then  $f$  is open.

The recent result Theorem B is a substantial improvement of Theorem A'. To improve Theorem B we consider the following conditions (O2) (=p11) and (O3) for  $f : X \rightarrow Y$ .

(O2)  $\forall x \in X$  and  $\varepsilon > 0 \exists \delta > 0$  such that  $f(x) + f(U_\delta) \subset f(x + U_\varepsilon)$ .

(O3)  $\exists \delta > 0$  and  $C \geq 1$  such that if  $\{x_j\} \subset X$  for which  $\sum_{j=1}^{\infty} \|x_j\| < \delta$  and  $\sum_{j=1}^{\infty} f(x_j)$  converges, then  $\sum_{j=1}^{\infty} f(x_j) = f(u)$  with  $\|u\| \leq C \sum_{j=1}^{\infty} \|x_j\|$ .

The conditions (O1), (O2) and (O3) are independent of either continuity or linearity of concerned mappings.

**Example 2.1** Let  $f_0 : \mathbb{R} \rightarrow \mathbb{R}$  be the function in Example 1.4 (1).  $f_0$  is nonlinear and everywhere discontinuous but (O1), (O2) and (O3) hold for  $f_0$ . In fact, for every  $r > 0$ ,  $\overline{f_0(U_r)} = f_0(U_r) = f_0[(-r, r)] = \mathbb{R} \in \mathcal{N}(\mathbb{R})$ ; for every  $x \in \mathbb{R}$  and  $\varepsilon > 0$ ,  $f_0(x + U_\varepsilon) = f_0[(x - \varepsilon, x + \varepsilon)] = \mathbb{R} = f_0(x) + f_0[(-\delta, \delta)] = f_0(x) + f_0(U_\delta)$  for all  $\delta > 0$ ; if  $0 < \sum_{j=1}^{\infty} |x_j| < +\infty$  and  $\sum_{j=1}^{\infty} f_0(x_j)$  converges, then  $\sum_{j=1}^{\infty} f_0(x_j) \in \mathbb{R} = f_0[(0, \sum_{j=1}^{\infty} |x_j|)]$  and so  $\sum_{j=1}^{\infty} f_0(x_j) = f_0(u)$  with  $0 < u < \sum_{j=1}^{\infty} |x_j|$ .

The existing open mapping theorems are related to continuity and linearity of concerned mappings, e.g., Theorem A and Theorem A' deal with continuous linear operators, Theorem B handles mappings which are continuous at 0, etc. However, using the classical Banach-Schauder method we have the following version which is independent of either continuity or linearity of concerned mappings.

**Theorem 2.1** Let  $X, Y$  be Hausdorff paranormed spaces. If  $f : X \rightarrow Y$  satisfies (O1), (O2) and (O3), then  $f$  is open. Conversely, if  $f : X \rightarrow Y$  is open and  $f(0) = 0$ , then (O1) holds for  $f$ .

Proof. Let  $\delta > 0$  and  $C \geq 1$  as in (O3),  $0 < r < \delta$ . By (O1),  $\overline{f(U_{r/(2^k C)})} \in \mathcal{N}(Y)$  for all  $k$  and so there exist integers  $n_1 < n_2 < n_3 < \dots$  such that

$V_{1/n_k} = \{v \in Y : \|v\| < 1/n_k\} \subset \overline{f(U_{r/(2^k C)})}$  for all  $k$ . Let  $y \in \overline{f(U_{r/(2C)})}$  and pick an  $x_1 \in U_{r/(2C)}$  for which  $\|y - f(x_1)\| < 1/n_2$ . Then  $y - f(x_1) \in V_{1/n_2} \subset \overline{f(U_{r/(2^2 C)})}$  and so there is an  $x_2 \in U_{r/(2^2 C)}$  such that  $y - f(x_1) - f(x_2) \in V_{1/n_3} \subset \overline{f(U_{r/(2^3 C)})}$ . Continuing this construction we have  $\{x_j\} \subset X$  such that

$$\|x_j\| < r/(2^j C) \text{ for all } j, \quad \|y - \sum_{j=1}^k f(x_j)\| < 1/n_{k+1} \text{ for all } k.$$

Then  $\sum_{j=1}^{\infty} \|x_j\| < r/C \leq r < \delta$  and  $\sum_{j=1}^{\infty} f(x_j) = y$ . By (O3),  $\sum_{j=1}^{\infty} f(x_j) = f(u)$  with  $\|u\| \leq C \sum_{j=1}^{\infty} \|x_j\| < r$  and  $y = f(u) \in f(U_r)$  since  $Y$  is Hausdorff. This shows that  $\overline{f(U_{r/(2C)})} \subset f(U_r)$  and, by (O1),  $f(U_r) \in \mathcal{N}(Y)$  whenever  $0 < r < \delta$ . Hence  $f(U_r) \in \mathcal{N}(Y)$  for all  $r > 0$ .

Let  $G$  be an open set in  $X$  and  $x \in G$ . Then  $x + U_\varepsilon \subset G$  for some  $\varepsilon > 0$ , and  $f(x) = f(x + 0) \in f(x + U_\varepsilon)$ . By (O2),  $f(x) + f(U_\delta) \subset f(x + U_\varepsilon) \subset f(G)$  for some  $\delta > 0$  but  $f(U_\delta) \in \mathcal{N}(Y)$  and so  $f(x)$  is an interior point of  $f(G)$ , hence  $f(G)$  is open.

Conversely, if  $f : X \rightarrow Y$  is open and  $f(0) = 0$ , then  $f(U_r)$  is open and  $0 = f(0) \in f(U_r) \subset \overline{f(U_r)}$  for all  $r > 0$ , i.e., (O1) holds for  $f$ .  $\square$

The condition (O3) looks like (pl4) but they are quite different. In fact, (pl4) implies that  $\lim_{x \rightarrow 0} f(x) = f(0) = 0$  but (O3) can hold for some everywhere discontinuous mappings.

The everywhere discontinuous open mapping  $f_0 : \mathbb{R} \rightarrow \mathbb{R}$  [12, p.168] satisfies (O1), (O2) and (O3) (Example 2.1) and so Theorem 2.1 is available for  $f_0$ .

Since (O3) is actually weaker than (pl4), we have a substantial improvement of Theorem B as follows.

**Corollary 2.1** *Let  $X, Y$  be Hausdorff paranormed spaces and  $f : X \rightarrow Y$  such that  $f(0) = 0$  and  $f(X)$  is of second category in  $Y$ . If (pl1), (pl2), (pl3) and (O3) hold for  $f$ , then  $f$  is open.*

Recall that  $0\text{-}pa(X, Y)$  is a large extension of the family of linear operators.

**Theorem 2.2** *Let  $X, Y$  be Hausdorff paranormed spaces and  $f \in 0\text{-}pa(X, Y)$ . If  $f$  is open, then both (O1) and (O3) hold for  $f$ .*

Proof. By Proposition 1.2,  $f(0) = 0$ . Then (O1) holds for  $f$  by Theorem 2.1.

Since  $f \in 0\text{-}pa(X, Y) = \bigcup_{\delta > 0} \delta\text{-}pa(X, Y)$ ,  $f \in \delta\text{-}pa(X, Y)$  for some  $\delta > 0$ . Assume that  $\{x_j\} \subset X$  for which  $0 < \sum_{j=1}^{\infty} \|x_j\| < \delta$  and  $\sum_{j=1}^{\infty} f(x_j)$  converges. Pick a  $k \in \mathbb{N}$  such that  $\frac{1}{k} < \min\{\delta - \sum_{j=1}^{\infty} \|x_j\|, \frac{1}{100} \sum_{j=1}^{\infty} \|x_j\|\}$ . Since  $\lim_n \sum_{j=n}^{\infty} f(x_j) = 0$  and  $f(U_{1/k}) \in \mathcal{N}(Y)$ ,  $\sum_{j=n_0}^{\infty} f(x_j) \in f(U_{1/k})$  for some  $n_0 > 1$ , i.e.,  $\sum_{j=n_0}^{\infty} f(x_j) = f(w)$  for some  $w \in U_{1/k}$ . Then  $\|w\| < \frac{1}{k} < \delta - \sum_{j=1}^{\infty} \|x_j\|$  and  $\sum_{j=1}^{\infty} f(x_j) = \sum_{j=1}^{n_0-1} f(x_j) + \sum_{j=n_0}^{\infty} f(x_j) = \sum_{j=1}^{n_0-1} f(x_j) + f(w)$ . By Proposition 1.4,  $\sum_{j=1}^{n_0-1} f(x_j) = f(z)$  with  $\|z\| \leq \sum_{j=1}^{n_0-1} \|x_j\|$ . Then  $\|z\| + \|w\| < \sum_{j=1}^{n_0-1} \|x_j\| + \delta - \sum_{j=1}^{\infty} \|x_j\| \leq \delta$  and  $\sum_{j=1}^{\infty} f(x_j) = f(z) + f(w) = f(u)$  for some  $u \in X$  with  $\|u\| \leq \|z\| + \|w\| < \sum_{j=1}^{n_0-1} \|x_j\| + \frac{1}{k} < \sum_{j=1}^{n_0-1} \|x_j\| + \frac{1}{100} \sum_{j=1}^{\infty} \|x_j\| \leq (1 + \frac{1}{100}) \sum_{j=1}^{\infty} \|x_j\|$ . Thus, (O3) holds for  $f$ .  $\square$

By Theorem 2.2, (O3) is a very important object for the open mapping problem. The following simple fact is also interesting.

**Lemma 2.1** *For  $f : X \rightarrow Y$  and  $x_0 \in X$  let  $F(x) = f(x_0 + x) - f(x_0)$ ,  $x \in X$ . Then  $F(0) = 0$ ;  $f$  is continuous at  $x_0$  if and only if  $F$  is continuous at 0;  $f$  is open if and only if  $F$  is open.*

We now are interested in the family  $\{f \in Y^X : f \text{ is continuous at some } x_0 \in X\}$ . In view of Lemma 2.1, it is enough to consider the family  $\{f \in Y^X : \lim_{x \rightarrow 0} f(x) = f(0) = 0\}$ . Observe that there is  $f \in pa(X, Y)$  for which  $\lim_{x \rightarrow 0} f(x) = f(0) = 0$  but  $f$  is discontinuous at each  $x \neq 0$  (Example 1.4 (3)).

**Theorem 2.3** *Let  $X, Y$  be Hausdorff paranormed spaces and  $f \in 0\text{-}pa(X, Y)$  which is continuous at 0. Then  $f$  is open if and only if (O1), (O2) and (O3) hold for  $f$ .*

Proof. By Theorem 2.1 and Theorem 2.2, we only need to show that if  $f$  is

open, then (O2) holds for  $f$ . Suppose that  $f$  is open and  $x \in X$ ,  $\varepsilon > 0$ . By Proposition 1.2,  $f(0) = 0$ . Since  $f(x + U_\varepsilon)$  is open and  $f(x) = f(x + 0) \in f(x + U_\varepsilon)$ , there is a  $V \in \mathcal{N}(Y)$  such that  $f(x) + V \subset f(x + U_\varepsilon)$  but  $\lim_{z \rightarrow 0} f(z) = f(0) = 0$  and so  $f(U_\delta) \subset V$  for some  $\delta > 0$ . Thus,  $f(x) + f(U_\delta) \subset f(x + U_\varepsilon)$ , (O2) holds for  $f$ .  $\square$

Theorem A' requires the domain space  $X$  is complete. Without completeness, we have

**Corollary 2.2** *Let  $X$  and  $Y$  be Hausdorff paranormed spaces. A continuous linear operator  $f : X \rightarrow Y$  is open if and only if (O1) and (O3) hold for  $f$ .*

We show that Theorem 2.3 is available for many nonlinear mappings.

**Example 2.2** *Let  $\Omega$  be a compact Hausdorff space,  $C(\Omega) = \{\xi \in \mathbb{R}^\Omega : \xi \text{ is continuous}\}$ . Let  $\varphi \in pa(\mathbb{R}, \mathbb{R})$  be a nonlinear homeomorphism from  $\mathbb{R}$  onto  $\mathbb{R}$ , e.g.,  $t^3$ ,  $\sinh t$ ,*

$$\varphi(t) = \begin{cases} e^t - 1, & t \geq 0, \\ 1 - e^{-t}, & t < 0, \end{cases}$$

*etc. Define  $f_\varphi : C(\Omega) \rightarrow C(\Omega)$  by  $f_\varphi(\xi) = \varphi \circ \xi$ ,  $\xi \in C(\Omega)$ . By Proposition 1.8,  $f_\varphi \in pa(C(\Omega), C(\Omega))$ . Let  $\xi_n \rightarrow 0$  in  $C(\Omega)$ , i.e.,  $\lim_n \sup_{\omega \in \Omega} |\xi_n(\omega)| = \lim_n \|\xi_n\| = 0$ . Let  $\varepsilon > 0$ . By Proposition 1.2,  $\varphi(0) = 0$  and there is a  $\delta > 0$  such that  $|\varphi(t)| < \varepsilon$  whenever  $|t| < \delta$ . Pick an  $n_0 \in \mathbb{N}$  for which  $\sup_{\omega \in \Omega} |\xi_n(\omega)| < \delta$  for  $n > n_0$ . Then  $\|\varphi \circ \xi_n\| = \sup_{\omega \in \Omega} |\varphi(\xi_n(\omega))| \leq \varepsilon$  for all  $n > n_0$ . This shows that  $\|\varphi \circ \xi_n\| \rightarrow 0$  and so  $f_\varphi(\xi_n) = \varphi \circ \xi_n \rightarrow 0$ , i.e.,  $f_\varphi$  is continuous at  $0 \in C(\Omega)$ .*

*We show that (O1), (O2) and (O3) hold for  $f_\varphi$ . Let  $r > 0$ . Since  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is a homeomorphism with  $\varphi(\mathbb{R}) = \mathbb{R}$  and  $\varphi(0) = 0$ , there is a  $\delta > 0$  such that  $|\varphi^{-1}(t)| < \frac{r}{2}$  for all  $|t| < \delta$ . For  $\xi \in U_\delta = \{\xi \in C(\Omega) : \|\xi\| < \delta\}$  and  $\omega \in \Omega$ ,  $|\xi(\omega)| < \delta$ ,  $|\varphi^{-1}(\xi(\omega))| < \frac{r}{2}$ ,  $\|\varphi^{-1} \circ \xi\| = \sup_{\omega \in \Omega} |\varphi^{-1}(\xi(\omega))| \leq \frac{r}{2} < r$  and so  $\xi = \varphi \circ \varphi^{-1} \circ \xi = f_\varphi(\varphi^{-1} \circ \xi) \in f_\varphi(U_r)$ . Thus,  $U_\delta \subset f_\varphi(U_r)$  and so  $f_\varphi(U_r) \in \mathcal{N}(C(\Omega))$  for all  $r > 0$  and, in particular, (O1) holds for  $f_\varphi$ .*

Let  $\xi \in C(\Omega)$ ,  $0 < \varepsilon < 1$  and  $M = \|\xi\| + \|\varphi \circ \xi\| + 1$ . Since  $\varphi^{-1} : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, there is a  $\theta \in (0, 1)$  such that  $|\varphi^{-1}(t) - \varphi^{-1}(s)| < \frac{\varepsilon}{2}$  whenever  $t, s \in [-M, M]$  with  $|t - s| < \theta$ . Then pick a  $\delta > 0$  for which  $|\varphi(t)| < \theta$  whenever  $|t| < \delta$ . If  $\eta \in U_\delta$  then  $|\eta(\omega)| < \delta$ ,  $|\varphi(\eta(\omega))| < \theta < 1$  for all  $\omega \in \Omega$  and  $|\varphi^{-1}[\varphi(\xi(\omega)) + \varphi(\eta(\omega))] - \xi(\omega)| = |\varphi^{-1}[\varphi(\xi(\omega)) + \varphi(\eta(\omega))] - \varphi^{-1}[\varphi(\xi(\omega))]| < \frac{\varepsilon}{2}$  for all  $\omega \in \Omega$ , i.e.,  $\|\varphi^{-1} \circ (\varphi \circ \xi + \varphi \circ \eta) - \xi\| \leq \frac{\varepsilon}{2} < \varepsilon$ ,  $\varphi^{-1} \circ (\varphi \circ \xi + \varphi \circ \eta) \in \xi + U_\varepsilon$  and  $f_\varphi(\xi) + f_\varphi(\eta) = \varphi \circ \xi + \varphi \circ \eta = \varphi \circ \varphi^{-1} \circ (\varphi \circ \xi + \varphi \circ \eta) = f_\varphi[\varphi^{-1} \circ (\varphi \circ \xi + \varphi \circ \eta)] \in f_\varphi(\xi + U_\varepsilon)$ . Thus,  $f_\varphi(\xi) + f_\varphi(U_\delta) \subset f_\varphi(\xi + U_\varepsilon)$ , (O2) holds for  $f_\varphi$ .

Although there is a somewhat complicated argument to show that (O3) holds for  $f_\varphi$ , we would like to derive (O3) from Theorem 2.2. Since (O2) holds for  $f_\varphi$  and  $f_\varphi(U_r) \in \mathcal{N}(C(\Omega))$  for all  $r > 0$ ,  $f_\varphi$  is open but  $f_\varphi \in pa(C(\Omega), C(\Omega))$  so Theorem 2.2 shows that (O3) holds for  $f_\varphi$ .

We show that either (O2) or (O3) is independent of the pseudo-additivity.

**Example 2.3** (1) Let

$$f(x) = \begin{cases} x, & x \in Q, \\ 0, & x \in \mathbb{R} \setminus Q. \end{cases}$$

Then  $f \in pa(\mathbb{R}, \mathbb{R})$  but both (O2) and (O3) fail to hold for  $f$ . In fact, for every  $\delta > 0$ ,  $f(\pi) + f[(-\delta, \delta)] = f[(-\delta, \delta)] = (-\delta, \delta) \cap Q \not\subset \{0\} \cup [(\pi - \frac{1}{2}, \pi + \frac{1}{2}) \cap Q] = f[(\pi - \frac{1}{2}, \pi + \frac{1}{2})] = f[\pi + (-\frac{1}{2}, \frac{1}{2})]$ , (O2) fails to hold for  $f$ . Let  $\delta > 0$  and  $x_0 \in (0, \delta) \setminus Q$ . Pick a strictly increasing  $\{r_j\} \subset Q$  such that  $r_1 = 0$  and  $r_j \rightarrow x_0$ . Then  $\sum_{j=1}^{\infty} |r_{j+1} - r_j| = \sum_{j=1}^{\infty} (r_{j+1} - r_j) = x_0 < \delta$  and  $\sum_{j=1}^{\infty} f(r_{j+1} - r_j) = \sum_{j=1}^{\infty} (r_{j+1} - r_j) = x_0$  but  $\sum_{j=1}^{\infty} f(r_{j+1} - r_j) \neq f(u)$  for all  $u \in \mathbb{R}$ : if  $u \in Q$ , then  $f(u) = u \neq x_0$ ; if  $u \in \mathbb{R} \setminus Q$ , then  $f(u) = 0 \neq x_0$ . Thus, (O3) fails to hold for  $f$ .

(2) Let  $f(x) = \sqrt[3]{x}$ ,  $x \in \mathbb{R}$ . Then (O2) holds for  $f$  but  $f \notin 0\text{-}pa(\mathbb{R}, \mathbb{R})$ . In fact, if  $\delta > 0$  then  $|\frac{\delta}{8}| + |\frac{\delta}{8}| = \frac{\delta}{4} < \delta$  but  $\sqrt[3]{\delta} = f(\frac{\delta}{8}) + f(\frac{\delta}{8}) = f(u) = \sqrt[3]{u}$  shows that  $u = \delta > |\frac{\delta}{8}| + |\frac{\delta}{8}|$ .

(3) Let  $f(x) = \arctan x$ ,  $x \in \mathbb{R}$ . Let  $\{x_j\} \subset \mathbb{R}$  with  $0 < \sum_{j=1}^{\infty} |x_j| < \frac{\pi}{4}$ .

Then  $\sum_{j=1}^{\infty} |\arctan x_j| \leq \sum_{j=1}^{\infty} |x_j| < \frac{\pi}{4}$ ,  $\sum_{j=1}^{\infty} f(x_j) = \sum_{j=1}^{\infty} \arctan x_j = v$  with  $|v| \leq \sum_{j=1}^{\infty} |\arctan x_j| \leq \sum_{j=1}^{\infty} |x_j| < \frac{\pi}{4}$  and  $v = \arctan(\alpha v) = f(\alpha v)$  for some  $\alpha > 1$  but  $(\frac{\tan t}{t})' > 0$  whenever  $0 < t < \frac{\pi}{2}$  and so  $\alpha = \frac{\tan v}{v} = \frac{\tan |v|}{|v|} < \frac{\tan \frac{\pi}{4}}{\frac{\pi}{4}} = \frac{4}{\pi}$ . Thus,  $\sum_{j=1}^{\infty} f(x_j) = f(\alpha v)$  with  $|\alpha v| < \frac{4}{\pi} \sum_{j=1}^{\infty} |x_j|$ , (O3) holds for  $f$ . However,  $f \notin 0\text{-}pa(\mathbb{R}, \mathbb{R})$ .

We also need to show that (O3) is independent of (O2).

**Example 2.4** (1) (O2) holds for  $f(x) = \sqrt[3]{x}$  ( $x \in \mathbb{R}$ ) but (O3) fails to hold for  $f$ . To this end let  $\delta > 0$  and for each  $k \in \mathbb{N}$  pick an  $\alpha_k \in (0, \delta/k)$ . If  $kf(\alpha_k) = k\sqrt[3]{\alpha_k} = \sqrt[3]{u} = f(u)$ , then  $k\alpha_k < \delta$  but  $|u| = u = k^2(k\alpha_k)$ .

(2) Let

$$f(x) = \begin{cases} f_0(x), & x \leq 1, \\ x - 1, & x > 1, \end{cases}$$

where  $f_0$  as in Example 2.1. Then (O3) holds for  $f$  but (O2) fails to hold for  $f$ : for every  $\delta > 0$ ,  $f(2) + f[(-\delta, \delta)] = 1 + f[(-\delta, \delta)] = \mathbb{R} \not\subset (\frac{1}{2}, \frac{3}{2}) = f[(\frac{3}{2}, \frac{5}{2}]] = f[2 + (-\frac{1}{2}, \frac{1}{2})]$ .

### 3 Special open mappings

**Lemma 3.1** Let  $X, Y$  be paranormed spaces and  $f : X \rightarrow Y$  for which 0 is an interior point of  $f(X)$ . If  $f(x) + f(z) \in f(X)$  for all  $x, z \in X$ , then  $f(X) = Y$ . In particular,  $f(X) = Y$  for every open  $f \in pa(X, Y)$ .

Proof.  $V \subset f(X)$  for some  $V \in \mathcal{N}(Y)$ . Let  $y \in Y$ . Then  $\frac{1}{n}y \in V \subset f(X)$  for some  $n > 2$ . Say that  $\frac{1}{n}y = f(x)$ . Then  $y = nf(x)$  but there is  $\{u_1, u_2, \dots, u_{n-1}\} \subset X$  such that  $2f(x) = f(x) + f(x) = f(u_1)$ ,  $3f(x) = f(u_1) + f(x) = f(u_2)$ ,  $\dots$ ,  $nf(x) = f(u_{n-2}) + f(x) = f(u_{n-1})$ , hence  $y = nf(x) = f(u_{n-1}) \in f(X)$ .  $\square$

By Theorem 2.1, if  $X, Y$  are Hausdorff paranormed spaces and (O1), (O2), (O3) hold for  $f : X \rightarrow Y$  but  $f(X) \neq Y$ , then  $f \notin pa(X, Y)$ .

**Example 3.1** Let  $(X, \|\cdot\|)$  be a nontrivial Banach space and

$$h_0(x) = \begin{cases} 0, & x = 0, \\ \frac{\arctan \|x\|}{\|x\|}x, & x \in X \setminus \{0\}. \end{cases}$$

It is easy to see that  $h_0 : (X, \|\cdot\|) \rightarrow (X, \|\cdot\|)$  is open and  $h_0(X) = U_{\pi/2} = \{x \in X : \|x\| < \frac{\pi}{2}\} \neq X$  so  $h_0 \notin pa(X, X) = \bigcap_{\delta > 0} \delta\text{-}pa(X, X)$ . Moreover,  $h_0 \notin 0\text{-}pa(X, X) = \bigcup_{\delta > 0} \delta\text{-}pa(X, X)$ . In fact, if  $\delta > 0$ ,  $0 < \|x\| < \min(\delta/2, \pi/4)$  and  $\frac{2\arctan \|x\|}{\|x\|}x = h_0(x) + h_0(x) = h_0(u) = \frac{\arctan \|u\|}{\|u\|}u$ , then  $2\arctan \|x\| = \arctan \|u\|$  and so  $\|u\| > \|x\| + \|x\|$ , i.e.,  $h_0 \notin \delta\text{-}pa(X, X)$  for all  $\delta > 0$ .

Note that (O1), (O2) and (O3) hold for  $h_0$ , though  $h_0 \notin 0\text{-}pa(X, X)$ .

Henceforth,  $(X, \|\cdot\|)$  is a nontrivial Banach space and  $U_r = \{x \in X : \|x\| < r\}$  for  $r > 0$ . According to Example 3.1 we introduce

**Definition 3.1** A subset  $S$  of  $(X, \|\cdot\|)$  is said to be a condensation of  $(X, \|\cdot\|)$  if the following (co1), (co2) and (co3) hold for  $S$ .

(co1)  $\sup_{x \in S} \|x\| < +\infty$ .

(co2) There exist operations  $\boxplus : S^2 \rightarrow S$  and  $*$  :  $\mathbb{K} \times S \rightarrow S$  such that  $[S] = (S; \boxplus, *)$  is a vector space and there is a norm  $||| \cdot ||| : [S] \rightarrow [0, +\infty)$  such that  $([S], ||| \cdot |||)$  is isometrically isomorphic with  $(X, \|\cdot\|)$ .

(co3) Letting  $d(x, z) = \|x - z\|$  for  $x, z \in X$ ,  $([S], d)$  is linearly homeomorphic with  $(X, d) = (X, \|\cdot\|)$ .

**Lemma 3.2** Let  $E \neq \emptyset$  and  $f : (X, \|\cdot\|) \rightarrow E$  be one to one. Then there exist operations  $\boxplus : f(X) \times f(X) \rightarrow f(X)$  and  $*$  :  $\mathbb{K} \times f(X) \rightarrow f(X)$  such that  $[f(X)] = (f(X); \boxplus, *)$  is a vector space and there is a norm  $||| \cdot ||| : [f(X)] \rightarrow [0, +\infty)$  such that  $(X, \|\cdot\|)$  is isometrically isomorphic with  $([f(X)], ||| \cdot |||)$  via  $f$ .



Proof. For  $x, z \in X$  and  $t \in \mathbb{K}$  let

$$f(x) \boxplus f(z) = f(x + z), \quad t * f(x) = f(tx) \text{ and } |||f(x)||| = \|x\|.$$

Then  $f(tx + sz) = f(tx) \boxplus f(sz) = t * f(x) \boxplus s * f(z)$ ,  $\forall x, z \in X, t, s \in \mathbb{K}$ ,  $f : X \rightarrow [f(X)]$  is linear, one to one and onto. Since  $f(x) \boxplus f(0) = f(x + 0) = f(x)$ ,  $f(0)$  is the zero vector of  $[f(X)]$ .

Since  $|||f(x)||| = \|x\|$  and  $f$  is one to one,  $|||f(x)||| = 0$  if and only if  $x = 0$  if and only if  $f(x) = f(0)$ ;  $|||t * f(x)||| = |||f(tx)||| = \|tx\| = |t|\|x\| = |t||||f(x)|||$ ;  $|||f(x) \boxplus f(z)||| = |||f(x + z)||| = \|x + z\| \leq \|x\| + \|z\| = |||f(x)||| + |||f(z)|||$ . Hence  $||| \cdot |||$  is a norm on  $[f(X)]$ .

We write  $f(x) \boxplus (-1) * f(z) = f(x) \boxminus f(z)$ . Then  $f(x) \boxminus f(z) = f(x - z)$  and  $|||f(x) \boxminus f(z)||| = |||f(x - z)||| = \|x - z\|$  for all  $x, z \in X$ .  $\square$

**Theorem 3.1** *Let  $f : (X, \|\cdot\|) \rightarrow (X, \|\cdot\|)$  be continuous, one to one and  $\sup_{x \in X} \|f(x)\| < +\infty$ . If (O1), (O2) and (O3) hold for  $f$ , then  $f(X)$  is a condensation of  $(X, \|\cdot\|)$ , and  $f \notin pa(X, X)$ .*

Proof. With the notations in Lemma 3.2,  $(X, \|\cdot\|)$  is isometrically isomorphic with  $([f(X)], ||| \cdot |||)$  via  $f$ .

By Theorem 2.1,  $f : (X, \|\cdot\|) \rightarrow (X, \|\cdot\|)$  is open. Let  $d(x, z) = \|x - z\|$  for  $x, z \in X$ . Then  $(f^{-1})^{-1}(G) = f(G) = f(G) \cap f(X)$  is open in  $(f(X), d)$  whenever  $G$  is open in  $(X, d)$ , i.e.,  $f^{-1} : (f(X), d) \rightarrow (X, d)$  is continuous and so

$$\|x_n - x\| \rightarrow 0 \iff d(f(x_n), f(x)) = \|f(x_n) - f(x)\| \rightarrow 0.$$

If  $d(f(x_n), f(x)) \rightarrow 0$  and  $d(f(z_n), f(z)) \rightarrow 0$ , then  $\|x_n - x\| \rightarrow 0$ ,  $\|z_n - z\| \rightarrow 0$  and so  $\|x_n + z_n - (x + z)\| \rightarrow 0$ ,  $d(f(x_n) \boxplus f(z_n), f(x) \boxplus f(z)) = d(f(x_n + z_n), f(x + z)) \rightarrow 0$ . Moreover, if  $d(f(x_n), f(x)) \rightarrow 0$  and  $t_n \rightarrow t$  in  $\mathbb{K}$ , then  $\|x_n - x\| \rightarrow 0$ ,  $\|t_n x_n - tx\| \rightarrow 0$  and so  $d(t_n * f(x_n), t * f(x)) = d(f(t_n x_n), f(tx)) \rightarrow 0$ . Thus with the metric  $d(f(x), f(z)) = \|f(x) - f(z)\|$ ,  $([f(X)], d)$  is a metric linear space, and  $(X, d) = (X, \|\cdot\|)$  is linearly homeomorphic with  $([f(X)], d)$

via  $f$ . □

**Example 3.2**  $h_0 : (X, \|\cdot\|) \rightarrow (X, \|\cdot\|)$  as in Example 3.1. Then  $U_{\pi/2} = h_0(X)$  is a condensation of  $(X, \|\cdot\|)$ .

**Theorem 3.2** Every bounded open ball of a Banach space  $(X, \|\cdot\|)$  is a condensation of  $(X, \|\cdot\|)$ .

Proof. Let  $\varepsilon > 0$ ,  $x_0 \in X$  and  $h_0 : X \rightarrow X$  as in Example 3.1. Define  $f : X \rightarrow X$  by  $f(x) = x_0 + \frac{2\varepsilon}{\pi}h_0(x)$ ,  $x \in X$ . Then  $f$  is continuous, one to one and  $f(X) = x_0 + U_\varepsilon = \{x \in X : \|x - x_0\| < \varepsilon\}$ . For every open  $G \subset X$ ,  $f(G) = \{x_0 + \frac{2\varepsilon}{\pi}h_0(x) : x \in G\} = x_0 + \frac{2\varepsilon}{\pi}h_0(G)$  is open since  $h_0$  is open, i.e.,  $f$  is open. Thus,  $\|x_n - x\| \rightarrow 0 \iff d(f(x_n), f(x)) = \|f(x_n) - f(x)\| \rightarrow 0$ .

For  $u, v \in U_\varepsilon$  and  $t \in \mathbb{K}$  let

$$(x_0 + u) \boxplus (x_0 + v) = x_0 + \frac{2\varepsilon}{\pi}h_0[h_0^{-1}(\frac{\pi}{2\varepsilon}u) + h_0^{-1}(\frac{\pi}{2\varepsilon}v)],$$

$$t * (x_0 + u) = x_0 + \frac{2\varepsilon}{\pi}h_0[th_0^{-1}(\frac{\pi}{2\varepsilon}u)].$$

Then  $(x_0 + u) \boxplus x_0 = (x_0 + u) \boxplus (x_0 + 0) = x_0 + \frac{2\varepsilon}{\pi}h_0[h_0^{-1}(\frac{\pi}{2\varepsilon}u) + 0] = x_0 + u$  for all  $u \in U_\varepsilon$  and  $[x_0 + U_\varepsilon] = (x_0 + U_\varepsilon; \boxplus, *)$  forms a vector space with the zero vector  $x_0$ . Let

$$|||x_0 + u||| = \|h_0^{-1}(\frac{\pi}{2\varepsilon}u)\| \text{ for } u \in U_\varepsilon, \quad d(x, z) = \|x - z\| \text{ for } x, z \in X.$$

Then  $f$  is an isometric isomorphism from  $(X, \|\cdot\|)$  onto  $([x_0 + U_\varepsilon], |||\cdot|||)$  and  $(X, d) = (X, \|\cdot\|)$  is linearly homeomorphic with  $([x_0 + U_\varepsilon], d)$  via  $f$ . □

For the condensation  $x_0 + U_\varepsilon$  in the above proof, the metric linear space  $([x_0 + U_\varepsilon], d)$  is linearly homeomorphic with the Banach space  $(X, \|\cdot\|)$  but  $([x_0 + U_\varepsilon], d)$  is not complete. In fact, for every  $x \in X \setminus \{0\}$ ,  $\{x_0 + \frac{2\varepsilon}{\pi}h_0(nx)\}_{n=1}^\infty$

is Cauchy in  $([x_0 + U_\varepsilon], d)$ :

$$\begin{aligned}
& d(x_0 + \frac{2\varepsilon}{\pi}h_0(nx), x_0 + \frac{2\varepsilon}{\pi}h_0(mx)) \\
&= d(x_0 + \frac{2\varepsilon}{\pi} \frac{\arctan \|nx\|}{\|nx\|} nx, x_0 + \frac{2\varepsilon}{\pi} \frac{\arctan \|mx\|}{\|mx\|} mx) \\
&= \frac{2\varepsilon}{\pi} \left| \arctan(n\|x\|) - \arctan(m\|x\|) \right| \rightarrow 0 \text{ as } n, m \rightarrow +\infty.
\end{aligned}$$

However,  $\{x_0 + \frac{2\varepsilon}{\pi}h_0(nx)\}$  is not convergent in  $([x_0 + U_\varepsilon], d)$  because  $\|f^{-1}[x_0 + \frac{2\varepsilon}{\pi}h_0(nx)]\| = \|nx\| \rightarrow +\infty$ , where  $f : (X, \|\cdot\|) \rightarrow ([x_0 + U_\varepsilon], d)$  is the linear homeomorphism in the proof of Theorem 3.2. This shows that  $d(\cdot, \cdot)$  is not an invariant metric on  $[x_0 + U_\varepsilon]$ , i.e., there exist  $u, v, w \in U_\varepsilon$  such that  $d(x_0 + u, x_0 + v) \neq d((x_0 + u) \boxplus (x_0 + w), (x_0 + v) \boxplus (x_0 + w))$ . In fact, for the case of  $X = \mathbb{R}$ , if  $x_0 = 0$ ,  $\varepsilon = \frac{\pi}{2}$ ,  $u = w = h_0(1) = \arctan 1$  and  $v = h_0(-1) = -\arctan 1$ , then  $u \boxplus w = h_0(2) = \arctan 2$ ,  $v \boxplus w = h_0(0) = 0$  and so  $d(u, v) = |h_0(1) - h_0(-1)| = 2 \arctan 1 = \frac{\pi}{2} > \arctan 2 = |h_0(2) - h_0(0)| = d(u \boxplus w, v \boxplus w)$ .

**Corollary 3.1** *Every nonempty open set in  $(X, \|\cdot\|)$  is a union of condensations of  $(X, \|\cdot\|)$ .*

*Proof.* Let  $G$  be a nonempty open subset of  $(X, \|\cdot\|)$ . For each  $x \in G$  there is an  $\varepsilon(x) > 0$  such that  $x + U_{\varepsilon(x)} \subset G$ . Then  $G = \bigcup \{x + U_{\varepsilon(x)} : x \in G\}$ .  $\square$

For  $A \subset X$ , let  $D(A) = \sup_{x, z \in A} \|x - z\|$ , the diameter of  $A$ .

**Definition 3.2** *A mapping  $f : X \rightarrow X$  is called an eyeshot to  $(X, \|\cdot\|)$  if the following (e1), (e2), (e3) and (e4) hold for  $f$ .*

(e1)  *$f$  is continuous and one to one.*

(e2)  *$\forall x \in X \exists t \in (0, 1)$  such that  $f(x) = tx$ .*

(e3) *If  $x, z \in X$  and  $\|x\| \leq \|z\|$ , then  $\|f(x)\| \leq \|f(z)\|$ .*

(e4)  *$\lim_{\|x\| \rightarrow +\infty} D[f(x + U_r)] = 0, \forall r > 0$ .*

We say that  $f : X \rightarrow X$  is a proper eyeshot to  $(X, \|\cdot\|)$  if (e1), (e2), (e3) and the following (e5) hold for  $f$ .

$$(e5) \sup_{x \in X} \|f(x)\| < +\infty.$$

**Example 3.3** (1) *Let*

$$f(x) = \begin{cases} \frac{1}{2}\|x\|x, & \|x\| \leq 1, \\ \frac{1}{2} \frac{x}{\sqrt{\|x\|}}, & \|x\| > 1. \end{cases}$$

*It is easy to see that  $f$  is an eyeshot to  $(X, \|\cdot\|)$  but  $f$  is not a proper eyeshot to  $(X, \|\cdot\|)$ .*

(2) *Let*

$$f(x) = \begin{cases} 0, & x = 0, \\ \frac{\tanh \|x\|}{\|x\|}x, & x \in X \setminus \{0\}. \end{cases}$$

*Then (e1), (e2) and (e3) hold for  $f$ . Moreover,  $\|f(x)\| = \tanh \|x\|$  and so  $\sup_{x \in X} \|f(x)\| = 1$ , (e5) holds for  $f$ ,  $f$  is a proper eyeshot to  $(X, \|\cdot\|)$ .*

**Lemma 3.3** *If  $f : X \rightarrow X$  satisfies (e1), (e2) and (e3), then*

$$(1) f(0) = 0, f(x) = \frac{\|f(x)\|}{\|x\|}x, \forall x \in X \setminus \{0\};$$

(2)  $\|f(x)\| = \|f(z)\|$  if and only if  $\|x\| = \|z\|$ ;  $\|f(x)\| < \|f(z)\|$  if and only if  $\|x\| < \|z\|$ ;

$$(3) \text{ if } z \in f(X) \text{ then } \{x \in X : \|x\| \leq \|z\|\} \subset f(X);$$

(4) if  $\sup_{x \in X} \|f(x)\| = +\infty$ , then  $f(X) = X$ ; if  $\Lambda = \sup_{x \in X} \|f(x)\| < +\infty$ , then  $\Lambda > 0$  and  $f(X) = U_\Lambda$ .

Proof. By (e2),  $f(0) = t0 = 0$  and  $f(x) = t_x x$  with  $0 < t_x < 1$ ,  $\forall x \in X \setminus \{0\}$ . For  $x \in X \setminus \{0\}$ ,  $\|f(x)\| = t_x \|x\|$ ,  $t_x = \frac{\|f(x)\|}{\|x\|}$ ,  $f(x) = \frac{\|f(x)\|}{\|x\|}x$ .

By (e3), if  $\|x\| = \|z\|$  then  $\|f(x)\| = \|f(z)\|$ . Assume that  $\|f(x)\| = \|f(z)\|$  but  $\|x\| \neq \|z\|$ . If  $x = 0$ , then  $\|f(z)\| = \|f(x)\| = \|f(0)\| = \|0\| = 0$ , hence  $f(z) = 0$ . But  $f$  is one to one and  $f(0) = 0$ . So  $z = 0$ ,  $\|x\| = 0 = \|z\|$ . Say that  $0 < \|x\| < \|z\|$ . Then  $\|x\| = \alpha\|z\| = \|\alpha z\|$  with  $\alpha = \frac{\|x\|}{\|z\|} \in (0, 1)$  and  $\|f(\alpha z)\| = \|f(x)\| = \|f(z)\|$ . Since  $f(\alpha z) = t_{\alpha z}$  and  $f(z) = sz$  where

$t, s \in (0, 1)$ ,  $t\alpha\|z\| = \|f(\alpha z)\| = \|f(z)\| = s\|z\|$  and so  $t\alpha = s$ ,  $f(\alpha z) = f(z)$ . But  $f$  is one to one so  $\alpha z = z$ ,  $\alpha = 1 \notin (0, 1)$ . This contradiction shows that  $\|f(x)\| = \|f(z)\|$  implies  $\|x\| = \|z\|$ .

If  $\|x\| < \|z\|$  then  $\|f(x)\| \leq \|f(z)\|$  by (e3) and so  $\|f(x)\| < \|f(z)\|$ . If  $\|f(x)\| < \|f(z)\|$  then  $\|x\| < \|z\|$  because  $\|z\| \leq \|x\|$  implies  $\|f(z)\| \leq \|f(x)\|$ .

Let  $0 \neq z \in f(X)$ . Then  $z = f(u) = tu$  for some  $u \in X \setminus \{0\}$  and  $0 < t < 1$ , hence  $u = \frac{1}{t}z$  and  $z = f(\frac{1}{t}z)$ . Let  $0 < \|x\| \leq \|z\|$ . Since  $\|\frac{1}{t}z\| = \|\frac{\|z\|}{t\|x\|}x\|$ ,  $\|z\| = \|f(\frac{1}{t}z)\| = \|f(\frac{\|z\|}{t\|x\|}x)\|$  and  $f(\frac{\|z\|}{t\|x\|}x) = \gamma\frac{\|z\|}{t\|x\|}x$  with  $0 < \gamma < 1$ ,  $\|z\| = \frac{\gamma}{t}\|z\|$ ,  $\gamma = t$  and  $f(\frac{\|z\|}{t\|x\|}x) = \frac{\|z\|}{\|x\|}x$ . Define  $\xi : [0, \frac{\|z\|}{t\|x\|}] \rightarrow [0, +\infty)$  by  $\xi(0) = 0$ ,  $\xi(\alpha) = s$  if  $f(\alpha x) = sx$ . In fact, if  $0 < \alpha \leq \frac{\|z\|}{t\|x\|}$  then there exists  $r \in (0, 1)$  such that  $f(\alpha x) = r\alpha x$  and so  $\xi(\alpha) = r\alpha$ . If  $\alpha_n \rightarrow \alpha$  in  $[0, \frac{\|z\|}{t\|x\|}]$ , then  $\alpha_n x \rightarrow \alpha x$ ,  $\xi(\alpha_n)x = f(\alpha_n x) \rightarrow f(\alpha x) = \xi(\alpha)x$  and so  $\xi(\alpha_n) \rightarrow \xi(\alpha)$ , i.e.,  $\xi$  is continuous. But  $\xi(0) = 0 < 1 \leq \frac{\|z\|}{\|x\|} = \xi(\frac{\|z\|}{t\|x\|})$  and so  $1 = \xi(\alpha)$  for some  $0 < \alpha \leq \frac{\|z\|}{t\|x\|}$ ,  $x = \xi(\alpha)x = f(\alpha x) \in f(X)$ , (3) holds for  $f$ .

If  $\sup_{x \in X} \|f(x)\| = +\infty$  and  $x \in X$ , then  $\|f(u)\| > \|x\|$  for some  $u \in X$  and so  $x \in f(X)$  by (3). Thus,  $f(X) = X$ .

Suppose that  $\Lambda = \sup_{x \in X} \|f(x)\| < +\infty$ . Since  $f(0) = 0$  and  $f$  is one to one,  $f(x) \neq 0$  whenever  $x \in X \setminus \{0\}$  and so  $\Lambda > 0$ . If  $0 < \|x\| < \Lambda$ , then  $\|f(u)\| > \|x\|$  for some  $u \in X$  and so  $x \in f(X)$  by (3), i.e.,  $U_\Lambda \subset f(X)$ . If  $\|f(x_0)\| = \Lambda$  for some  $x_0 \in X$ , then  $x_0 \neq 0$ ,  $\|x_0\| < \|2x_0\|$  and so  $\Lambda = \|f(x_0)\| < \|f(2x_0)\| \leq \Lambda$ . This contradiction shows that  $\{x \in X : \|x\| = \Lambda\} \subset X \setminus f(X)$ . Thus,  $f(X) = U_\Lambda$ .  $\square$

**Lemma 3.4** *A mapping  $f : X \rightarrow X$  is a proper eyeshot to  $(X, \|\cdot\|)$  if and only if*

(i)  $f(0) = 0$  and

(ii) *there is a continuous  $\xi : (0, +\infty) \rightarrow (0, 1)$  such that  $t\xi(t)$  is strictly*

increasing in  $(0, +\infty)$ ,  $\sup_{t>0} t\xi(t) < +\infty$  and

$$f(x) = \xi(\|x\|)x, \quad \forall x \in X \setminus \{0\}.$$

Proof.  $\implies$  By Lemma 3.3,  $f(0) = 0$ . Pick a nonzero  $u \in X$  and let  $x_0 = \frac{1}{\|u\|}u$  and  $\xi(t) = \frac{\|f(tx_0)\|}{t}$  for  $t > 0$ . Then  $\|x_0\| = 1$  and for every  $z_0 \in X$  with  $\|z_0\| = 1$ ,  $\|f(tz_0)\| = \|f(tx_0)\|$  for all  $t \in (0, +\infty)$  by Lemma 3.3. Thus,  $\xi$  is independent of the choice of  $\|x_0\| = 1$ . Since  $\|f(tx_0)\| = \|\alpha_t tx_0\| = t\alpha_t$  with  $0 < \alpha_t < 1$ ,  $\xi(t) = \alpha_t \in (0, 1)$  for all  $t > 0$ .  $\xi$  is continuous since  $f$  is continuous.

By Lemma 3.3, it follows from  $\|x\| = \|\|x\|x_0\|$  that  $\|f(x)\| = \|f(\|x\|x_0)\|$  and  $f(x) = \frac{\|f(x)\|}{\|x\|}x = \frac{\|f(\|x\|x_0)\|}{\|x\|}x = \xi(\|x\|)x$  for  $x \in X \setminus \{0\}$ .

If  $0 < t < s < +\infty$ , then  $\|tx_0\| = t < s = \|sx_0\|$  and  $t\xi(t) = \|f(tx_0)\| < \|f(sx_0)\| = s\xi(s)$  by Lemma 3.3. Moreover,  $t\xi(t) = \|f(tx_0)\| \leq \sup_{x \in X} \|f(x)\| < +\infty$  for all  $t > 0$ .

$\Leftarrow$  Since  $0 < \xi(t) < 1$  for all  $t > 0$ , if  $x_n \rightarrow 0$  in  $X$  and  $x_n \neq 0$  for all  $n$ , then  $f(x_n) = \xi(\|x_n\|)x_n \rightarrow 0 = f(0)$ . If  $x_n \rightarrow x \neq 0$ , then  $x_n \neq 0$  eventually and  $\|x_n\| \rightarrow \|x\|$ ,  $\xi(\|x_n\|) \rightarrow \xi(\|x\|)$ ,  $f(x_n) = \xi(\|x_n\|)x_n \rightarrow \xi(\|x\|)x = f(x)$ . Thus,  $f$  is continuous.

Since  $\xi(t) \neq 0$  for all  $t > 0$ ,  $f(x) = \xi(\|x\|)x \neq 0 = f(0)$  for all  $x \neq 0$ . If  $x, z \in X \setminus \{0\}$  and  $f(x) = f(z)$ , then  $\xi(\|x\|)x = \xi(\|z\|)z$ ,  $\xi(\|x\|)\|x\| = \xi(\|z\|)\|z\|$ . By (ii),  $\|x\| = \|z\|$  and  $\xi(\|x\|) = \xi(\|z\|) > 0$ , hence  $x = z$ . Thus,  $f$  is one to one.

Let  $x, z \in X$  with  $\|x\| \leq \|z\|$ . If  $x = 0$  then  $\|f(x)\| = \|0\| = 0 \leq \|f(z)\|$ . If  $0 < \|x\| \leq \|z\|$ , then  $\|f(x)\| = \xi(\|x\|)\|x\| \leq \xi(\|z\|)\|z\| = \|f(z)\|$  by (ii).

It follows from (ii) that  $\|f(x)\| = \|x\|\xi(\|x\|) \leq \sup_{t>0} t\xi(t) < +\infty$ ,  $\forall x \in X \setminus \{0\}$ .  $\square$

**Theorem 3.3** *Every proper eyeshot to  $(X, \|\cdot\|)$  is an eyeshot to  $(X, \|\cdot\|)$ .*

Proof. Let  $f : X \rightarrow X$  be a proper eyeshot to  $(X, \|\cdot\|)$ . By Lemma 3.4,  $f(0) = 0$  and  $f(x) = \xi(\|x\|)x$  for  $x \neq 0$ , where  $\xi \in (0, 1)^{(0, +\infty)}$  is continuous,  $t\xi(t)$  is strictly increasing in  $(0, +\infty)$ ,  $0 < \sup_{t>0} t\xi(t) = A < +\infty$ . Clearly,  $\lim_{t \rightarrow +\infty} t\xi(t) = A$ .

Let  $r > 0$ . If  $u, v \in U_r$  and  $\|x\| > r$ , then

$$\begin{aligned}
& \|f(x+u) - f(x+v)\| \\
&= \|\xi(\|x+u\|)(x+u) - \xi(\|x+v\|)(x+v)\| \\
&= \|[\xi(\|x+u\|) - \xi(\|x+v\|)]x + \xi(\|x+u\|)u - \xi(\|x+v\|)v\| \\
&\leq \left| \xi(\|x+u\|) - \xi(\|x+v\|) \right| \|x\| + \xi(\|x+u\|)\|x+u\| \frac{\|u\|}{\|x+u\|} \\
&\quad + \xi(\|x+v\|)\|x+v\| \frac{\|v\|}{\|x+v\|} \\
&\leq \left| \xi(\|x+u\|)\|x+u\| \frac{\|x\|}{\|x+u\|} - \xi(\|x+u\|)\|x+u\| \frac{\|x\|}{\|x+v\|} \right. \\
&\quad \left. + \xi(\|x+u\|)\|x+u\| \frac{\|x\|}{\|x+v\|} - \xi(\|x+v\|)\|x+v\| \frac{\|x\|}{\|x+v\|} \right| + \frac{2Ar}{\|x\| - r} \\
&\leq \xi(\|x+u\|)\|x+u\| \left| \frac{\|x\|}{\|x+u\|} - \frac{\|x\|}{\|x+v\|} \right| + \frac{\|x\|}{\|x+v\|} \left| \xi(\|x+u\|)\|x+u\| \right. \\
&\quad \left. - \xi(\|x+v\|)\|x+v\| \right| + \frac{2Ar}{\|x\| - r} \\
&\leq A \left| \frac{\|x\|}{\|x+u\|} - \frac{\|x\|}{\|x+v\|} \right| + \frac{\|x\|}{\|x\| - r} \left| \xi(\|x+u\|)\|x+u\| \right. \\
&\quad \left. - \xi(\|x+v\|)\|x+v\| \right| + \frac{2Ar}{\|x\| - r}.
\end{aligned}$$

Since  $1 - \frac{\|x\|}{\|x+u\|} \leq 1 - \frac{\|x\|}{\|x\|+r} = \frac{r}{\|x\|+r}$  and  $\frac{\|x\|}{\|x+u\|} - 1 \leq \frac{\|x\|}{\|x\|-r} - 1 = \frac{r}{\|x\|-r}$  for all  $u \in U_r$ ,  $\lim_{\|x\| \rightarrow +\infty} \frac{\|x\|}{\|x+u\|} = 1$  uniformly for  $u \in U_r$ . Moreover,  $\xi(\|x\| - r)(\|x\| - r) \leq \xi(\|x+u\|)\|x+u\| \leq \xi(\|x\| + r)(\|x\| + r)$  for all  $u \in U_r$  and so  $\lim_{\|x\| \rightarrow +\infty} \xi(\|x+u\|)\|x+u\| = A$  uniformly for  $u \in U_r$ . Then

$$\lim_{\|x\| \rightarrow +\infty} D[f(x+U_r)] = \lim_{\|x\| \rightarrow +\infty} \sup_{u, v \in U_r} \|f(x+u) - f(x+v)\| = 0.$$

□

**Lemma 3.5** *A linear operator  $f : X \rightarrow X$  satisfies (e1), (e2) and (e3) if and only if there is an  $\alpha \in (0, 1)$  such that  $f(x) = \alpha x$ ,  $\forall x \in X$ .*

Proof. Let  $f : X \rightarrow X$  be a linear operator such that (e1), (e2) and (e3) hold for  $f$ . Let  $x \in X \setminus \{0\}$ . By (e2),  $f(x) = \alpha x$  for some  $\alpha \in (0, 1)$ . Then  $f(0) = 0 = \alpha 0$ . If  $z \in X \setminus \{0\}$ , then  $f(z) = sz$  for some  $s \in (0, 1)$  but  $\|\frac{\|x\|}{\|z\|}z\| = \|x\|$  and  $s\|x\| = \frac{\|x\|}{\|z\|}s\|z\| = \frac{\|x\|}{\|z\|}\|f(z)\| = \|f(\frac{\|x\|}{\|z\|}z)\| = \|f(x)\| = \alpha\|x\|$  by Lemma 3.3 and the linearity of  $f$ . Hence  $s = \alpha$ ,  $f(z) = \alpha z$ .  $\square$

**Theorem 3.4** *If  $f : X \rightarrow X$  is a nonzero linear operator, then (e4) fails to hold for  $f$  and so  $f$  is not an eyeshot to  $(X, \|\cdot\|)$ .*

Proof. Since  $f \neq 0$ ,  $f(x_0) \neq 0$  for some  $x_0 \in X \setminus \{0\}$ . Let  $r > 0$ . Since  $\|\frac{r}{2\|x_0\|}x_0\| = \frac{r}{2} < r$ ,  $D[f(x + U_r)] = \sup\{\|f(x + u) - f(x + v)\| : u, v \in U_r\} = \sup\{\|f(u - v)\| : u, v \in U_r\} \geq \|f(\frac{r}{2\|x_0\|}x_0 - 0)\| = \frac{r}{2\|x_0\|}\|f(x_0)\|$  for all  $x \in X$ , where  $\frac{r}{2\|x_0\|}\|f(x_0)\| > 0$ . Hence (e4) fails to hold for  $f$ .  $\square$

**Lemma 3.6** *Let  $x \in X$  and  $0 < \varepsilon < \|x\|$ . Then*

$$\{\alpha u : \alpha > 0, u \in X, \|u - x\| < \frac{\varepsilon}{4}, \|x\| - \frac{\varepsilon}{4} < \|\alpha u\| < \|x\| + \frac{\varepsilon}{4}\} \subset x + U_\varepsilon.$$

Proof. Let  $\|u - x\| < \frac{\varepsilon}{4}$ ,  $\alpha > 0$  and  $\|x\| - \frac{\varepsilon}{4} < \|\alpha u\| < \|x\| + \frac{\varepsilon}{4}$ . Then

$$\|u\| = \|x - (x - u)\| \geq \|x\| - \|x - u\| = \|x\| - \|u - x\| > \varepsilon - \frac{\varepsilon}{4} = \frac{3\varepsilon}{4} > 0.$$

Since  $\|x\| - \frac{\varepsilon}{4} < \alpha\|u\| < \|x\| + \frac{\varepsilon}{4}$ ,  $\frac{\|x\| - \varepsilon/4}{\|u\|} < \alpha < \frac{\|x\| + \varepsilon/4}{\|u\|}$  and so

$$\alpha - 1 < \frac{\|x\| + \frac{\varepsilon}{4}}{\|u\|} - 1 = \frac{\|x\| + \frac{\varepsilon}{4} - \|u\|}{\|u\|} \leq \frac{\|x - u\| + \frac{\varepsilon}{4}}{\|u\|} < \frac{\varepsilon}{2\|u\|},$$

$$1 - \alpha < 1 - \frac{\|x\| - \frac{\varepsilon}{4}}{\|u\|} = \frac{\|u\| - \|x\| + \frac{\varepsilon}{4}}{\|u\|} \leq \frac{\|u - x\| + \frac{\varepsilon}{4}}{\|u\|} < \frac{\varepsilon}{2\|u\|},$$

i.e.,  $|\alpha - 1| < \frac{\varepsilon}{2\|u\|}$ . Then

$$\|\alpha u - x\| \leq \|\alpha u - u\| + \|u - x\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{4} < \varepsilon.$$

$\square$

By Theorem 2.3, if a continuous linear operator  $f : X \rightarrow X$  is open, then (O1), (O2) and (O3) hold for  $f$ . Then we would like to show that if  $f : X \rightarrow X$



is an eyeshot to  $(X, \|\cdot\|)$ , then  $f$  is a nonlinear open mapping but (O1) and (O2) hold for  $f$ .

Recall the notations  $[f(X)] = (f(X); \boxplus, *)$  in Lemma 3.2.

**Theorem 3.5** *If  $f : X \rightarrow X$  is an eyeshot to  $(X, \|\cdot\|)$ , then (O1) and (O2) hold for  $f$  and  $f$  is a nonlinear open mapping but  $f$  is a linear homeomorphism from  $(X, d) = (X, \|\cdot\|)$  onto  $([f(X)], d)$ , where  $d(x, z) = \|x - z\|$  for  $x, z \in X$ .*

Proof. Let  $r > 0$ . Pick an  $x_0 \in X$  for which  $\|x_0\| = r$ , e.g.,  $x_0 = \frac{r}{\|x\|}x$  where  $x \neq 0$ . Then  $f(x_0) = tx_0$  for some  $t \in (0, 1)$  and  $\|f(x_0)\| = tr$ . By Lemma 3.3, if  $\|x\| < r = \|x_0\|$  then  $\|f(x)\| < \|f(x_0)\| = tr$ , hence  $f(U_r) \subset U_{tr}$ . By Lemma 3.3 again, if  $\|x\| < tr = \|f(x_0)\|$  then  $x = f(u)$  with  $\|u\| < \|x_0\| = r$ , hence  $U_{tr} \subseteq f(U_r)$ . Thus,  $f(U_r) = U_{tr} \in \mathcal{N}(X)$ , i.e.,  $\forall r > 0 \exists 0 < \delta < r$  such that  $f(U_r) = U_\delta \in \mathcal{N}(X)$  and, in particular, (O1) holds for  $f$ .

We claim that if  $0 < r < \sup_{x \in X} \|f(x)\|$  then  $U_r = f(U_\delta)$  for some  $\delta > r$ . To this end let  $0 < r < \sup_{x \in X} \|f(x)\|$  and pick an  $x_0 \in X$  with  $\|x_0\| = r$ . By Lemma 3.3,  $x_0 \in f(X)$ ,  $x_0 = f(u) = tu$  for some  $u \in X$  and  $t \in (0, 1)$ , hence  $u = \frac{1}{t}x_0$  and  $x_0 = f(\frac{1}{t}x_0)$ . If  $\|x\| < r = \|x_0\| = \|f(\frac{1}{t}x_0)\|$ , then Lemma 3.3 shows that  $x = f(v)$  with  $\|v\| < \|\frac{1}{t}x_0\| = \frac{r}{t}$ , hence  $U_r \subseteq f(U_{r/t})$ . Conversely, if  $x = f(v)$  with  $\|v\| < r/t = \|\frac{1}{t}x_0\|$ , then  $\|x\| = \|f(v)\| < \|f(\frac{1}{t}x_0)\| = \|x_0\| = r$ , hence  $f(U_{r/t}) \subseteq U_r$ . Thus,  $U_r = f(U_{r/t})$  with  $\frac{r}{t} > r$ .

We show that (O2) holds for  $f$ . Let  $x \in X$  and  $\varepsilon > 0$ . If  $x = 0$ , then  $f(x) + f(U_\varepsilon) = f(0) + f(U_\varepsilon) = 0 + f(U_\varepsilon) = f(U_\varepsilon) = f(0 + U_\varepsilon) = f(x + U_\varepsilon)$ . Suppose that  $x \neq 0$ ,  $0 < \varepsilon_0 < \min(\varepsilon, \|x\|, \sup_{y \in X} \|f(y)\|)$  and

$$A = \{\alpha u : \alpha > 0, u \in X, \|u - x\| < \frac{\varepsilon_0}{4}, \|x\| - \frac{\varepsilon_0}{4} < \|\alpha u\| < \|x\| + \frac{\varepsilon_0}{4}\}.$$

If  $\|u - x\| < \frac{\varepsilon_0}{4}$  then  $\|u\| = \|x - (x - u)\| \geq \|x\| - \|x - u\| > \varepsilon_0 - \frac{\varepsilon_0}{4} > 0$  and so  $0 \notin A$ . By Lemma 3.6,  $A \subset x + U_{\varepsilon_0} \subset x + U_\varepsilon$ . If  $u \in x + U_{\varepsilon_0/4}$ , then  $\|u - x\| < \frac{\varepsilon_0}{4}$  and  $\|x\| - \frac{\varepsilon_0}{4} < \|x\| - \|x - u\| \leq \|x - (x - u)\| = \|u\| = \|x + u - x\| \leq \|x\| + \|u - x\| < \|x\| + \frac{\varepsilon_0}{4}$  and so  $u \in A$ . Thus,  $x + U_{\varepsilon_0/4} \subset A \subset x + U_{\varepsilon_0} \subset x + U_\varepsilon$ .

By (e2),  $f(x) = tx$  with  $0 < t < 1$ . Since  $1 - \frac{\varepsilon_0}{4\|x\|} > \frac{3}{4} > 0$ , there exist  $\alpha > 0$  and  $\beta > 0$  such that  $f[(1 - \frac{\varepsilon_0}{4\|x\|})x] = \alpha x$  and  $f[(1 + \frac{\varepsilon_0}{4\|x\|})x] = \beta x$ . By Lemma 3.3, it follows from  $\|(1 - \frac{\varepsilon_0}{4\|x\|})x\| = \|x\| - \frac{\varepsilon_0}{4} < \|x\| < \|x\| + \frac{\varepsilon_0}{4} = \|(1 + \frac{\varepsilon_0}{4\|x\|})x\|$  that  $\alpha\|x\| < t\|x\| < \beta\|x\|$ ,  $\alpha < t < \beta$ .

Let  $r = \min\{\frac{t\varepsilon_0}{4}, (t - \alpha)\|x\|, (\beta - t)\|x\|\}$  and  $z \in tx + U_r$ , i.e.,  $\|z - tx\| < r$ . Then  $t\|\frac{1}{t}z - x\| = \|z - tx\| < r \leq \frac{t\varepsilon_0}{4}$  and so  $\|\frac{1}{t}z - x\| < \frac{\varepsilon_0}{4}$ , i.e.,  $\frac{1}{t}z \in x + U_{\varepsilon_0/4}$ , and  $\|f[(1 - \frac{\varepsilon_0}{4\|x\|})x]\| = \alpha\|x\| = t\|x\| - (t - \alpha)\|x\| \leq \|tx\| - r < \|tx\| - \|tx - z\| \leq \|z\| \leq \|tx\| + \|z - tx\| < \|tx\| + r \leq t\|x\| + (\beta - t)\|x\| = \beta\|x\| = \|f[(1 + \frac{\varepsilon_0}{4\|x\|})x]\|$ . By Lemma 3.3,  $z \in f(X)$  and  $z = f(\gamma z)$  with  $\gamma > 1$  and so  $\|f[(1 - \frac{\varepsilon_0}{4\|x\|})x]\| < \|f(\gamma z)\| < \|f[(1 + \frac{\varepsilon_0}{4\|x\|})x]\|$ . By Lemma 3.3 again,  $\|x\| - \frac{\varepsilon_0}{4} = \|(1 - \frac{\varepsilon_0}{4\|x\|})x\| < \|\gamma z\| < \|(1 + \frac{\varepsilon_0}{4\|x\|})x\| = \|x\| + \frac{\varepsilon_0}{4}$ , it follows from  $\|\frac{1}{t}z - x\| < \frac{\varepsilon_0}{4}$  that  $\gamma z = \gamma t(\frac{1}{t}z) \in A$  and so  $z = f(\gamma z) \in f(A)$ , hence  $f(x) + U_r = tx + U_r \subset f(A) \subset f(x + U_{\varepsilon_0}) \subset f(x + U_\varepsilon)$ .

Since  $0 < r \leq \frac{t\varepsilon_0}{4} < \frac{\varepsilon_0}{4} < \varepsilon_0 < \sup_{y \in X} \|f(y)\|$ ,  $U_r = f(U_\delta)$  for some  $\delta > r$  and  $f(x) + f(U_\delta) = f(x) + U_r \subset f(x + U_\varepsilon)$ , hence (O2) holds for  $f$ .

Thus,  $f(U_r) \in \mathcal{N}(X)$  whenever  $r > 0$  and for every  $x \in X$  and  $\varepsilon > 0$  there is a  $\delta > 0$  such that  $f(x) + f(U_\delta) \subset f(x + U_\varepsilon)$ . This shows that  $f$  is open but  $f$  is not linear by Theorem 3.4.

Now the proof of Theorem 3.1 shows that  $f$  is a linear homeomorphism from  $(X, d) = (X, \|\cdot\|)$  onto  $([f(X)], d)$ .  $\square$

By Theorem 3.5, if  $f : X \rightarrow X$  is an eyeshot to  $(X, \|\cdot\|)$  then  $(X, d) = (X, \|\cdot\|)$  is linearly homeomorphic with  $([f(X)], d)$  via  $f$ . On the other hand,  $d(f(x), f(z)) = \|f(x) - f(z)\|$  is not an invariant metric on  $[f(X)]$ , i.e.,  $d(f(x) \boxplus f(w), f(z) \boxplus f(w)) \neq d(f(x), f(z))$  for some  $f(x), f(z), f(w) \in f(X)$ . We would like to show that this “disharmony” is very valuable for our eyesight.

**Example 3.4** Let  $X = \mathbb{R}^2$  and

$$f(x) = \begin{cases} 0, & x = 0, \\ \frac{2}{\pi} \frac{\arctan \|x\|}{\|x\|} x, & x \neq 0. \end{cases}$$

Then  $f$  is a proper eyeshot to  $\mathbb{R}^2$  and  $f(\mathbb{R}^2) = U_1 = \{(a, b) \in \mathbb{R}^2 : \sqrt{a^2 + b^2} < 1\}$ .

For each  $t \in \mathbb{R}$  let  $L_t$  be the line  $\{(a, a + t) : a \in \mathbb{R}\}$ . If  $t \neq s$  in  $\mathbb{R}$ , then the lines  $L_t$  and  $L_s$  are parallel in  $\mathbb{R}^2$  but  $\lim_{a \rightarrow +\infty} f[(a, a + t)] = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$  in  $\mathbb{R}^2$  for all  $t \in \mathbb{R}$ , where  $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) \notin f(\mathbb{R}^2)$ . Hence we would like to say that all parallel lines  $L_t$  ( $t \in \mathbb{R}$ ) meet at infinity. Moreover,  $\lim_{y \rightarrow +\infty} d((0, y^2), (y, y^2)) = \lim_{x \rightarrow +\infty} d((x, 0), (x, \sqrt{x})) = +\infty$  but  $\lim_{y \rightarrow +\infty} d(f((0, y^2)), f((y, y^2))) = \lim_{x \rightarrow +\infty} d(f((x, 0)), f((x, \sqrt{x}))) = 0$ .

$A = \{(a, b) \in \mathbb{R}^2 : b \geq 1\}$  is convex in  $\mathbb{R}^2$  but  $f(A) = \{\frac{2}{\pi} \frac{\arctan \sqrt{a^2 + b^2}}{\sqrt{a^2 + b^2}}(a, b) : a, b \in \mathbb{R}, b \geq 1\}$  is not convex in  $\mathbb{R}^2$ . However,  $f(A)$  is convex in the vector space  $[f(\mathbb{R}^2)] = (f(\mathbb{R}^2); \boxplus, *)$  and so we would like to say that  $f$  senses that  $f(A)$  is convex.

**Corollary 3.2** *If  $f : X \rightarrow X$  is a proper eyeshot to  $(X, \|\cdot\|)$ , then  $f(X)$  is a condensation of  $(X, \|\cdot\|)$ .*

Proof. By Lemma 3.3,  $0 < \Lambda = \sup_{x \in X} \|f(x)\| < +\infty$  and  $f(X) = U_\Lambda$ . By Theorem 3.2,  $f(X)$  is a condensation of  $(X, \|\cdot\|)$ .  $\square$

A map  $f : X \rightarrow Y$  is called closed, if the image of each closed set is closed, and a map that is simultaneously open and closed is called clopen [12, p.157; 13, p.45]. The everywhere discontinuous open mapping  $f_0 : \mathbb{R} \rightarrow \mathbb{R}$  (Example 1.4 (1)) is not closed [12, p.168].

**Corollary 3.3** *If  $f : X \rightarrow X$  is an eyeshot to  $(X, \|\cdot\|)$  but  $\sup_{x \in X} \|f(x)\| = +\infty$ , then  $f$  is a homeomorphism from  $(X, d)$  onto  $(X, d)$ , where  $d(x, z) = \|x - z\|$  for  $x, z \in X$  and, in particular,  $f : (X, \|\cdot\|) \rightarrow (X, \|\cdot\|)$  is clopen.*

Proof. By Lemma 3.3,  $f(X) = X$ . Theorem 3.5.  $\square$

**Corollary 3.4** *If  $f : X \rightarrow X$  is a proper eyeshot to  $(X, \|\cdot\|)$ , then  $f$  is a homeomorphism from  $(X, d)$  onto  $(f(X), d)$  and for every bounded closed subset  $B$  of  $(X, \|\cdot\|)$  the image  $f(B)$  is closed in  $(X, \|\cdot\|)$  but  $f : (X, \|\cdot\|) \rightarrow$*

$(X, \|\cdot\|)$  is not closed.

Proof.  $f(X) = U_\Lambda$  with  $0 < \Lambda < +\infty$ . By Theorem 3.5,  $f$  is a homeomorphism from  $(X, d) = (X, \|\cdot\|)$  onto  $(f(X), d)$ . Let  $B \subset X$  be closed and bounded in  $(X, \|\cdot\|)$ . Then there is a  $M > 0$  such that  $B \subset U_M$  and  $f(B) \subset f(U_M) = U_\delta$  for some  $0 < \delta < M$  (see the proof of Theorem 3.5). Since  $f$  is one to one, it follows from  $U_\delta = f(U_M) \subsetneq f(X) = U_\Lambda$  that  $\delta < \Lambda$  and so  $\overline{U_\delta} \subset U_\Lambda$ . If  $\{x_n\} \subset B$  and  $f(x_n) \rightarrow z$  in  $(X, \|\cdot\|)$ , then  $z \in \overline{U_\delta} \subset U_\Lambda$  and  $z = f(x)$  for some  $x \in X$ ,  $d(f(x_n), f(x)) = \|f(x_n) - z\| \rightarrow 0$ . By Theorem 3.5,  $\|x_n - x\| = d(x_n, x) \rightarrow 0$  and so  $x \in B$ ,  $z = f(x) \in f(B)$ . This shows that  $f(B)$  is closed in  $(X, \|\cdot\|)$ .

Let  $r > 0$ . Then  $X \setminus U_r$  is closed in  $(X, \|\cdot\|)$ . Since  $f(U_r) = U_\delta$  with  $0 < \delta < \min(r, \Lambda)$  and  $f$  is one to one,  $f(X \setminus U_r) = f(X) \setminus f(U_r) = U_\Lambda \setminus U_\delta$  and  $U_\Lambda \setminus U_\delta$  is not closed in  $(X, \|\cdot\|)$ .  $\square$

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